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Mit den herzlichsten Grüßen
von Georg Cantor.
Naunhof bei Leipzig, 13 April 1905

On the Quaternary Linear Homogeneous Groups Modulo p of Order a Multiple of p.

By Leonard Eugene Dickson.[*]

1. Consider the group G of all quaternary linear homogeneous transformations modulo p of determinant unity. The order of G is

$$\tau p^6, \qquad \tau \equiv (p^4-1)(p^3-1)(p^2-1). \tag{1}$$

Every subgroup of order a power of p is conjugate within G with a subgroup of the group G_{p^6} of the operators

$$[\alpha_{ij}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix}. \tag{2}$$

The subgroups of this G_{p^6} are given in §3. Some of these were determined in in an earlier paper[†], the others by similar methods.

Let H denote any subgroup of order $p^h N$ of G, N being prime to p, $h > 0$. Applying a suitable transformation within G, we may assume that H contains a subgroup G_{p^h} of G_{p^6}. If H contains G_{p^h} self-conjugately, H lies in the corresponding group determined in §4, and the explicit determination of H is essentially a problem on binary or ternary groups. Suppose next that G_{p^h} is not self-conjugate in H and let p^m be the maximal order of a subgroup common to G_{p^h} and any of its conjugates under H; let G_{p^m} be such a subgroup. By a theorem[‡] due to Burnside and Frobenius, H must contain an operator S, of period prime to p, commutative with G_{p^m} but not with G_{p^h}.

By §2, τ contains no factor of the form $1 + p^4 x$, $x > 0$. Hence the number of conjugates to G_{p^h} in H is not $\equiv 1 \pmod{p^4}$, so that[||] $m \overline{\overline{>}} h - 3$. Further, if

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[*] Research assistant to the Carnegie Institution of Washington for 1904.

[†] *American Journal of Mathematics*, vol. 27 (1905), pp. 280-302.

[‡] References in Burnside's *Theory of Groups*, p. 97.

[||] Compare Burnside's *Theory of Groups*, p. 94, Cor. II,

$p > 2$ and if the number of conjugates to G_{p^4} is not $(p^4-1)(p^3-1)$, we may (§2) set $m \geqq h-2$. To these results we owe the practicability of the method employed in the problem. The general theorem of §7 leads to a considerable reduction in the number of cases to be treated.

For $p=2$, G is simply isomorphic with the alternating group on 8 letters.* When exceptional, the case $p=2$ is excluded.

2. **LEMMA.** *The only factor of the form* $1+p^3x$, $x>0$, *of* τ *is* $(p^4-1)(p^3-1)$, *an additional factor* 9 *occurring if* $p=2$.

Let $\tau = (1+p^3x)\,q$. Then $p^2-1 \equiv q \pmod{p^3}$. Let $q = mp^3+p^2-1$, $m \geqq 0$. For $m=0$, we reach $(p^4-1)(p^3-1)$. Let henceforth $m \geqq 1$. Then

$$q>p^3,\quad 1+p^3x=\tau/q<(p^9-p^7)/p^3,\quad x<p^3-p.$$

Expanding $\tau-(1+p^3x)\,q$ and dividing by p^3, we get

$$p^6-p^4-p^3-p^2+p+1-mxp^3-p^2x+x-m=0.$$

Hence there is an integer t such that

$$m=x+p+1+tp^2. \tag{3}$$

Then $p^4-p^2-p-1-mxp-x-t=0$. Hence for some integer s,

$$x+t=sp-1,\quad p^3-p-1-mx-s=0. \tag{4}$$

If $t>0$, then $s>0$, $m>tp^2$, $q>tp^5$, $1+p^3x<p^4/t$, $x<p/t$. Hence $x<p$, $t<p$. Then by $(4)_1$, $sp-1 \leqq 2p-2$, $s=1$, $p-1<t+p/t$. If $p>3$, $(2t-p)^2 \geqq p^2-4p>(p-3)^2$, requiring $t=1$. If $p=3$, then $x=t=1$. In every case, $s=1$, $t=1$, $x=p-2$, $m=p^2+2p-1$, so that $(4)_2$ gives $p=1$.

If $t=0$, $x=sp-1$, $s>0$, $m=sp+p$. Then $(4)_2$ becomes

$$p^3-1-s^2p^2-sp^2+sp-s=0,$$

whence $p^3-s^2p^2>0$, $s=-1+wp \geqq p-1$. Thus $p=2$, $s=1$, $x=1$.

If $t=-r$, where $r>0$, (3) gives $x \geqq rp^2-p$. Then by $(4)_1$, $s>0$. But $x<p^3-p$. Hence $r<p$. Set $x=rp^2-p+z$, $z \geqq 0$. Then $m=z+1$, by (3), and $sp=rp^2-p+z+1-r$ by (4). Hence

$$z+1-r=lp,\quad s=rp-1+l.$$

---

* References and proof in *Linear Groups*, §268, p. 290.

By the first, $l \geqq 0$ since $r < p$. For $l = 0$, (4) becomes

$$p^3 - p = r^2p^3 + r^3 - r,$$

whence $r \equiv 0$ or $1 \pmod{p}$, $r = 1$, which is impossible. Let next $l \geqq 1$. Now $(4)_1$ may be written

$$p^3 - p - 1 = (r + lp)(rp^2 - p + lp + r - 1) + rp - 1 + l.$$

But the second member $> p\,(rp^2)$, the first member $< p^3$.

3. The $p^2 + p + 1$ subgroups of order p^5 of G_{p^6} are given by a linear homogeneous relation between α_{21}, α_{32}, α_{43}. For each G_{p^5} given in the first column of the following table, the subgroups of order p^4 are obtained by annexing a linear homogeneous relation between the elements in the second column:

$\{\alpha_{21} = 0\}$	α_{31}, α_{32}, α_{43}
$\{\alpha_{32} = 0\}$	α_{21}, α_{43}, α_{42}, α_{31}
$\{\alpha_{43} = 0\}$	α_{21}, α_{32}, α_{42}
$\{c_2\alpha_{21} + c_3\alpha_{32} + c_4\alpha_{43} = 0\}$	α_{21}, α_{32}, α_{43}, ϕ

where for the last G_{p^5}, at least two c_i are $\neq 0$, $p > 2$, and

$$\phi = c_2\alpha_{31} - c_4\alpha_{42} - \tfrac{1}{2}c_2\alpha_{32}\alpha_{21} + \tfrac{1}{2}c_4\alpha_{43}\alpha_{32}.$$

For $p > 2$, the $p^4 + 2p^3 + 2p^2 + p + 1$ distinct G_{p^4} are given in the first column of the following table; the subgroups of order p^3 of each G_{p^4} are obtained by annexing a linear relation between the elements in the second column:

$\{\alpha_{21} = 0,\ \alpha_{43} = 0\}$	α_{31}, α_{32}, α_{41}, α_{42}
$\{\alpha_{43} = 0,\ \alpha_{32} = s\alpha_{21}\}$	α_{21}, α_{42}, $a = \alpha_{31} - \frac{1}{2}s\alpha_{21}^2$
$\{\alpha_{21} = r\alpha_{43},\ \alpha_{32} = s\alpha_{43}\}$	α_{43}, $b = \alpha_{42} - \frac{1}{2}s\alpha_{43}^2$, $c = \alpha_{31} - \frac{1}{2}rs\alpha_{43}^2$
$\{\alpha_{21} = 0,\ \alpha_{31} = r\alpha_{32} + s\alpha_{43}\}$	α_{32}, α_{43}, $d = \alpha_{41} - r\alpha_{42} - \frac{1}{2}s\alpha_{43}^2$
$\{\alpha_{43} = 0,\ \alpha_{42} = r\alpha_{32} + s\alpha_{21}\}$	α_{21}, α_{32}, $e = \alpha_{41} - r\alpha_{31} - \frac{1}{2}s\alpha_{21}^2$
$\{\alpha_{32} = 0,\ \alpha_{31} = r\alpha_{21} + s\alpha_{43}\}$	α_{21}, α_{43}, α_{42}
$\{\alpha_{32} = 0,\ \alpha_{42} = r\alpha_{21} + s\alpha_{43} + t\alpha_{31}\}$	α_{21}, α_{43}, α_{31}
$\left\{\sum\limits_{i=2}^{4} c_i\alpha_{i\,i-1} = 0,\ \phi + \sum\limits_{i=2}^{4} k_i\alpha_{i\,i-1} = 0\right\}$	a_{21}, α_{32}, α_{43}

For $p > 2$, the $4p^4 + 3p^3 + 2p^2 + p + 1$ distinct G_{p^3} are given in the first column of the following table, the subgroups of order p^2 of each G_{p^3} are obtained by annexing a linear relation between the elements in the second column:

$\{\alpha_{21}=\alpha_{43}=0,\ \text{lin.}(\alpha_{31},\alpha_{32},\alpha_{41},\alpha_{42})=0\}$	$\alpha_{31},\ \alpha_{32},\ \alpha_{41},\ \alpha_{42}$
$\{\alpha_{43}=0,\ \alpha_{32}=s\alpha_{21},\ a=r\alpha_{21}+t\alpha_{43}\}$	$\alpha_{21},\ \alpha_{42}$
$\{\alpha_{43}=0,\ \alpha_{32}=s\alpha_{21},\ \alpha_{42}=r\alpha_{21}\}$	$\alpha_{21},\ a,\ f=\alpha_{41}-\frac{1}{2}r\alpha_{21}^2$
$\left\{\begin{matrix}\alpha_{21}=r\alpha_{43},\ \alpha_{32}=s\alpha_{43},\\ b=tc+u\alpha_{43}\end{matrix}\right\}$	α_{43}, c, if $rt\neq 1$, or $rt=1,p=3,s\neq 0$ α_{43}, c, g, if $rt=1$, with $s=0$ when $p=3$
$\left\{\begin{matrix}\alpha_{21}=r\alpha_{43},\ \alpha_{32}=s\alpha_{43},\\ c=t\alpha_{43}\end{matrix}\right\}$	α_{43}, b, if $r\neq 0$ α_{43}, b, $h=\alpha_{41}-\frac{1}{2}t\alpha^2_{43}$, if $r=0$.
$\left\{\begin{matrix}\alpha_{21}=0,\ \alpha_{31}=r\alpha_{32}+s\alpha_{43},\\ d=t\alpha_{32}+u\alpha_{43}\end{matrix}\right\}$	$\alpha_{32},\ \alpha_{43}$
$\left\{\begin{matrix}\alpha_{43}=0,\ \alpha_{42}=r\alpha_{32}+s\alpha_{21},\\ e=t\alpha_{21}+u\alpha_{32}\end{matrix}\right\}$	$\alpha_{21},\ \alpha_{32}$
$\left\{\begin{matrix}\alpha_{32}=0,\ \alpha_{31}=r\alpha_{21}+s\alpha_{43},\\ \alpha_{42}=t\alpha_{21}+u\alpha_{43}\end{matrix}\right\}$	α_{21}, α_{43}, if $r+u\neq 0$ α_{21}, α_{43}, j, if $r+u=0$

where the further abbreviations have been used

$$g=\alpha_{41}-\alpha_{31}\alpha_{43}+\tfrac{1}{3}rs\alpha_{43}^3-\tfrac{1}{2}ru\alpha_{43}^2,\ \ j=\alpha_{41}-\tfrac{1}{2}t\alpha_{21}^2-\tfrac{1}{2}s\alpha_{43}^2.$$

For $p>2$. the distinct G_{p^3} are given in the first column of the next table;

$$2p^5+3p^4+2p^3+2p^2+p+1\ (p>3),\ p^5+5p^4+p^3+2p^2+p+1\ (p=3)$$

is the number of G_{p^2}; the subgroups C_p of each G_{p^2} are obtained by annexing a linear relation between the elements of the second column:

$\{\alpha_{21}=\alpha_{43}=0,\ \text{two lin.}\ (\alpha_{31},\ \alpha_{32},\ \alpha_{41},\ \alpha_{42})\}$	$\alpha_{31},\ \alpha_{32},\ \alpha_{41},\ \alpha_{42}$
$\{\alpha_{43}=0,\ \alpha_{32}=s\alpha_{21},\ a=\rho\alpha_{21},\ \alpha_{42}=r\alpha_{21}\}$	$\alpha_{21},\ f$
$\{\alpha_{43}=0,\ \alpha_{32}=s\alpha_{21},\ \alpha_{42}=r\alpha_{21},\ f=ta+u\alpha_{21}\}$	$\alpha_{21},\ a$
$\left\{\begin{matrix}\alpha_{21}=r\alpha_{43},\ \alpha_{32}=s\alpha_{43},\ b=\rho\alpha_{43},\\ c=w\alpha_{43}\end{matrix}\right\}$	$\alpha_{43}=0$, if $p=3$, $rs\neq 0$ α_{43}, k, if $p>3$, or $p=3$, $rs=0$
$\left\{\begin{matrix}\alpha_{21}=r\alpha_{43},\ b=r^{-1}c+u\alpha_{43},\\ \alpha_{32}=s\alpha_{43},\ g=wc+z\alpha_{43},\end{matrix}\ s=0\text{ if }p=3\right\}$	$\alpha_{43},\ c$
$\{\alpha_{21}=0,\ \alpha_{32}=s\alpha_{43},\ \alpha_{31}=t\alpha_{43},\ h=wb+z\alpha_{43}\}$	$\alpha_{43},\ b$
$\left\{\begin{matrix}\alpha_{32}=0,\ \alpha_{31}=r\alpha_{21}+s\alpha_{43},\\ \alpha_{42}=t\alpha_{21}-r\alpha_{43},\ j=w\alpha_{21}+z\alpha_{43}\end{matrix}\right\}$	$\alpha_{21},\ \alpha_{43}$

where $k=\alpha_{41}-\frac{1}{6}rs\alpha_{43}^3-\frac{1}{2}(w+\rho r)\alpha_{43}^2$. The only distinct C_p in this table are the p^3+p^2+p+1 groups $\{\alpha_{21}=\alpha_{43}=0,\ \text{three lin.}\ (\alpha_{31},\ \alpha_{32},\ \alpha_{41},\ \alpha_{42})\}$, the p^4 groups

$$\{\alpha_{43}=0,\ \alpha_{32}=s\alpha_{21},\ a=\rho\alpha_{21},\ \alpha_{42}=r\alpha_{21},\ f=w\alpha_{21}\}$$

and the p^5 $(p > 3)$ or $5p^3$ $(p = 3)$ groups, with $rs = 0$ if $p = 3$,

$$\{\alpha_{21} = r\alpha_{43},\ \alpha_{32} = s\alpha_{43},\ b = \rho\alpha_{43},\ c = w\alpha_{43},\ k = z\alpha_{43}\}.$$

Hence there are $p^5 + p^4 + p^3 + p^2 + p + 1$ $(p > 3)$, $p^5 + p^2 + p + 1 = 256$ $(p = 3)$ distinct C_p in G_{p^6}, as is also evident directly.

4. In the first column of the following table appears a representative* of each set of conjugate subgroups of order a power of p of G, for $p > 2$. All quaternary transformations (δ_{ij}) commutative with this representative are given in the second column. The conditions for determinant unity are omitted for several reasons; they may be supplied by inspection. Here $\mu = 1$ or a particular not-square ν; $l = 0, 1, \nu$; ν_1 is any not-square.

G_{p^6}	$\delta_{ij} = 0\ (j > i)$
$\{\alpha_{s\,s-1} = 0\}$, $s = 2, 3$, or 4	$\delta_{ij} = 0\ (j > i)$ except $\delta_{s-1\,s}$
$\{\alpha_{21} = \alpha_{32} + \mu\alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{22}^2 = \delta_{11}\delta_{33}$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{21} = \mu\alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{11}\delta_{44} = \delta_{22}\delta_{33}$
$\{\alpha_{21} = \alpha_{32}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{22}^2 = \delta_{11}\delta_{33}$
$\{\alpha_{32} = \alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{21} = \alpha_{32} = 0\}$, $\{\alpha_{32} = \alpha_{43} = 0\}$	$\delta_{ij} = 0\ (j > i)$
$\{\alpha_{21} = \alpha_{43} = 0\}$	$\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0$
$\{\alpha_{21} = 0,\ \alpha_{32} = \alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{32} = 0,\ \alpha_{21} = \mu\alpha_{43}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{11}\delta_{44} = \delta_{22}\delta_{33}$
$\{\alpha_{43} = 0,\ \alpha_{21} = \alpha_{32}\}$	$\delta_{ij} = 0\ (j > i)$, $\delta_{22}^2 = \delta_{11}\delta_{33}$
$\{\alpha_{21} = \mu\alpha_{43},\ \alpha_{32} = \alpha_{43}\}$	“ $\delta_{22}^2 = \delta_{11}\delta_{33}$, $\delta_{33}^2 = \delta_{22}\delta_{44}$
$\{\alpha_{32} = 0,\ \alpha_{42} = \alpha_{21}\}$	“ $\delta_{32} = 0$, $\delta_{22}^2 = \delta_{11}\delta_{44}$
$\{\alpha_{32} = \alpha_{43},\ \alpha_{42} = \frac{1}{2}\alpha_{43}^2 + \alpha_{21}\}$	“ $\delta_{32}\delta_{44} = \delta_{43}\delta_{33}$, $\delta_{33}^2 = \delta_{22}\delta_{44}$, $\delta_{22}^2 = \delta_{11}\delta_{44}$
$\{\alpha_{32} = \alpha_{43},\ \alpha_{42} = \frac{1}{2}\alpha_{43}^2\}$	“ “ “
$\{\alpha_{21} = \alpha_{32},\ \alpha_{31} = \frac{1}{2}\alpha_{32}^2 + \alpha_{43}\}$	“ $\delta_{21}\delta_{33} = \delta_{32}\delta_{22}$, $\delta_{22}^2 = \delta_{11}\delta_{33}$, $\delta_{33}^2 = \delta_{11}\delta_{44}$
$\{\alpha_{21} = \alpha_{32},\ \alpha_{31} = \frac{1}{2}\alpha_{32}^2\}$	“ “ “
$\left\{\begin{matrix}\alpha_{21} = \mu\alpha_{43} \\ \alpha_{31} + \mu\alpha_{42} = \mu\alpha_{32}\alpha_{43} + \kappa\alpha_{43}\end{matrix}\right\}$	“ $\delta_{21}\delta_{33} = \mu\delta_{43}\delta_{11}$, $\delta_{22}\delta_{33} = \delta_{11}\delta_{44}$ also $\delta_{33}^2 = \delta_{11}\delta_{44}$ if $\kappa \neq 0$
$\left\{\begin{matrix}\alpha_{21} = \alpha_{32} + \mu\alpha_{43} \\ \alpha_{31} + \mu\alpha_{42} = \frac{1}{2}\alpha_{32}^2 + \mu\alpha_{32}\alpha_{43} + \kappa\alpha_{43}\end{matrix}\right\}$	“ $\delta_{21}\delta_{33} = \delta_{32}\delta_{22} + \mu\delta_{43}\delta_{11}$, $\delta_{22}\delta_{33} = \delta_{11}\delta_{44}$, $\delta_{22}^2 = \delta_{11}\delta_{33}$; also $\delta_{33} = \delta_{22}$ if $\kappa \neq 0$

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* I have aimed to list but one representative, but I have relied mostly upon inspection in this direction. However I have taken every precaution and check to make sure that every set of conjugates is represented in the list.

$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=0\}$	$\delta_{ij}=0\ (j>i)$
$\{\alpha_{21}=\alpha_{32}=\alpha_{31}=0\}$	$\delta_{14}=\delta_{24}=\delta_{34}=0$
$\{\alpha_{21}=\alpha_{32}=\alpha_{42}=0\}$	$\delta_{ij}=0\ (j>i),\ \delta_{32}=0$
$\{\alpha_{32}=\alpha_{43}=\alpha_{42}=0\}$	$\delta_{12}=\delta_{13}=\delta_{14}=0$
$\{\alpha_{21}=\alpha_{32}=0,\ \alpha_{31}=\alpha_{43}\}$	$\delta_{ij}=0\ (j>i),\ \delta_{33}^2=\delta_{11}\delta_{44}$
$\{\alpha_{21}=\alpha_{32}=0,\ \alpha_{31}=\mu\alpha_{42}\}$	“ $\delta_{32}=0,\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}$
$\{\alpha_{32}=\alpha_{43}=0,\ \alpha_{31}=\mu\alpha_{42}\}$	“ “ “
$\{\alpha_{32}=\alpha_{43}=0,\ \alpha_{42}=\alpha_{21}\}$	“ $\delta_{22}^2=\delta_{11}\delta_{44}$
$\{\alpha_{21}=\alpha_{43}=0,\ \alpha_{41}=\mu\alpha_{32}\}$	$\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0$ and (5)
$\left\{\begin{matrix}\alpha_{21}=0,\ \alpha_{32}=\alpha_{43}\\ \alpha_{42}-\frac{1}{2}\alpha_{43}^2=l\alpha_{31}\end{matrix}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{32}\delta_{44}\delta_{33}=\delta_{43}\delta_{33}^2+l\delta_{21}\delta_{44}^2.$ $\delta_{33}^2=\delta_{22}\delta_{44}$; also $\delta_{11}\delta_{44}=\delta_{22}\delta_{33}$ if $l\neq0$
$\left\{\begin{matrix}\alpha_{32}=0,\ \alpha_{21}=\mu\alpha_{43}\\ \alpha_{31}=-\mu\alpha_{42}+l\alpha_{43}\end{matrix}\right\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0,\ \delta_{22}\delta_{33}-\delta_{23}\delta_{32}=\delta_{11}\delta_{44}\ (l=0)$ $\delta_{ij}=0\ (j>i),\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}=\delta_{33}^2\ (l\neq0)$
$\left\{\begin{matrix}\alpha_{32}=0,\ \alpha_{21}=\mu\alpha_{43}\\ \alpha_{31}=m\alpha_{42},\ m\neq0,\ m\neq-\mu\end{matrix}\right\}$	“ , $\delta_{32}=0,\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}\ (m\neq\mu)$ Preceding (δ_{ij}) and $\delta_{23}\delta_{32}=\delta_{11}\delta_{44}$, $\delta_{12}=\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=\delta_{32}=\delta_{34}=0\ (m=\mu)$
$\left\{\begin{matrix}\alpha_{43}=0,\ \alpha_{21}=\alpha_{32}\\ \alpha_{31}-\frac{1}{2}\alpha_{32}^2=l\alpha_{42}\end{matrix}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{22}\delta_{33}=\delta_{33}\delta_{21}+l\delta_{11}\delta_{43}$, $\delta_{22}^2=\delta_{11}\delta_{33}$; also $\delta_{22}\delta_{33}=\delta_{11}\delta_{44}$ if $l\neq0$
$\{\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43},\ \alpha_{42}=\frac{1}{2}\alpha_{43}^2\}$	$\delta_{ij}=0\ (j>i),\ \delta_{44}\delta_{11}=\delta_{22}\delta_{33},\ \delta_{33}^2=\delta_{22}\delta_{44},\ \delta_{33}\delta_{43}=\delta_{44}\delta_{32}$
$\left\{\begin{matrix}\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43}\\ \alpha_{31}-\frac{1}{2}\mu\alpha_{43}^2=m\,(\alpha_{42}-\frac{1}{2}\alpha_{43}^2)\end{matrix}\right\}$	“ , $\delta_{33}^2=\delta_{22}\delta_{44},\ \delta_{22}^2=\delta_{11}\delta_{33}$, $\delta_{21}\delta_{44}=(\mu+m)\,\delta_{32}\delta_{33}-m\delta_{43}\delta_{22}$
$\left\{\begin{matrix}\alpha_{32}=0,\alpha_{42}=\alpha_{21},\alpha_{31}=q\alpha_{21}+m\alpha_{43}\\ q^2-4m \text{ a not-square}\end{matrix}\right\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0$ and (6)

$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=\alpha_{41}=0\}$	$\delta_{ij}=0\ (j>i),\ \delta_{21}=\delta_{43}=0$; prod. by (12)(34)
$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=\alpha_{42}=0\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0$
$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=\alpha_{31}=0\}$	$\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=\delta_{34}=0$
$\{\alpha_{21}=\alpha_{32}=\alpha_{43}=0,\ \alpha_{42}=\mu\alpha_{31}\}$	$\delta_{ij}=0\ (j>i),\ \delta_{11}\delta_{44}=\delta_{22}\delta_{33}$
$\{\alpha_{21}=\alpha_{32}=\alpha_{42}=0,\ \alpha_{31}=\alpha_{43}\}$	“ , $\delta_{32}=0,\ \delta_{33}^2=\delta_{11}\delta_{44}$
$\{\alpha_{21}=\alpha_{32}=0,\ \alpha_{31}=\alpha_{43},\ \alpha_{41}=\frac{1}{2}\alpha_{43}^2\}$	“ , $\delta_{21}=0,\ \delta_{33}^2=\delta_{11}\delta_{44},\ \delta_{11}\delta_{43}=\delta_{33}\delta_{31}$
$\{\alpha_{32}=\alpha_{43}=0,\ \alpha_{42}=\alpha_{21},\ \alpha_{41}=\frac{1}{2}\alpha_{21}^2\}$	“ , $\delta_{43}=0,\ \delta_{22}^2=\delta_{11}\delta_{44},\ \delta_{11}\delta_{42}=\delta_{22}\delta_{21}$
$\{\alpha_{21}=\alpha_{43}=0,\ \alpha_{41}=\alpha_{32},\ \alpha_{42}=\nu\alpha_{31}\}$	$\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0^*,\ \delta_{11}=\pm\delta_{22}$, $\delta_{12}=\pm\nu\delta_{21},\ \delta_{33}=\pm\delta_{44},\ \delta_{43}=\pm\nu\delta_{34}$
$\left\{\begin{matrix}\alpha_{32}=0,\ \alpha_{21}=\mu\alpha_{43}\\ \alpha_{31}=\mu\alpha_{42},\ \alpha_{41}=\mu\alpha_{42}\alpha_{43}\end{matrix}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{32}=0,\ \delta_{22}\delta_{33}=\delta_{11}\delta_{44}$ $\mu\delta_{43}\delta_{22}=\delta_{21}\delta_{44},\ \mu\delta_{42}\delta_{33}=\delta_{31}\delta_{44}$; prod. by (23)
$\left\{\begin{matrix}\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43},\\ \alpha_{42}=\frac{1}{2}\alpha_{43}^2,\ \alpha_{31}=\frac{1}{2}\mu\alpha_{43}^2\end{matrix}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{33}^2=\delta_{22}\delta_{44},\ \delta_{22}^2=\delta_{11}\delta_{33}$, $\delta_{43}\delta_{33}=\delta_{32}\delta_{44},\ \mu\delta_{32}\delta_{22}=\delta_{21}\delta_{33}$
$\left\{\begin{matrix}\alpha_{21}=\mu\alpha_{43},\ \alpha_{32}=\alpha_{43},\ (p>3)\\ \alpha_{31}=\mu\alpha_{42},\ \alpha_{41}=\mu\alpha_{43}\alpha_{42}-\frac{1}{3}\mu\alpha_{43}^3\end{matrix}\right\}$	$\delta_{ij}=0\ (j>i),\ \delta_{22}^2=\delta_{11}\delta_{33},\ \delta_{33}^2=\delta_{22}\delta_{44}$, $\delta_{33}\delta_{32}=\delta_{43}\delta_{22},\ \delta_{21}\delta_{44}=\mu\delta_{33}\delta_{32}$, $\delta_{42}\delta_{33}\delta_{44}+\delta_{43}\delta_{32}\delta_{44}-\delta_{43}^2\delta_{33}=\mu^{-1}\delta_{31}\delta_{44}^2$
$\left\{\begin{matrix}\alpha_{32}=0,\ \alpha_{42}=\alpha_{21},\ \alpha_{31}=-\nu_1\alpha_{43}\\ \alpha_{41}=\frac{1}{2}\alpha_{21}^2-\frac{1}{2}\nu_1\alpha_{43}^2\end{matrix}\right\}$	$\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0$, $\delta_{33}=\pm\delta_{22},\ \delta_{32}=\pm\nu_1\delta_{23}$, and (7)

* Footnote on opposite page.

$B_{41\alpha}: \xi_4' = \xi_4 + \alpha\xi_1$	$\delta_{12} = \delta_{13} = \delta_{14} = \delta_{24} = \delta_{34} = 0$
$B_{21\alpha}B_{43\mu\alpha}$	$\delta_{13} = \delta_{14} = \delta_{32} = \delta_{34} = 0,\ \delta_{11}\delta_{44} = \delta_{22}\delta_{33},$ $\delta_{11}\delta_{42} = \mu\delta_{23}\delta_{31},\ \delta_{22}\delta_{13} = \mu\delta_{11}\delta_{24}$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ \frac{1}{2}\alpha^2 & 0 & \alpha & 1 \end{pmatrix} = S_\alpha = S_1^{\alpha}$	$\delta_{ij} = 0\ (j > i),\ \delta_{32} = 0,$ $\delta_{33}^2 = \delta_{11}\delta_{44},\ \delta_{43}\delta_{33} = \delta_{31}\delta_{44},$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mu\alpha & 1 & 0 & 0 \\ \frac{1}{2}\mu\alpha^2 & \alpha & 1 & 0 \\ \frac{1}{6}\mu\alpha^3 & \frac{1}{2}\alpha^2 & \alpha & 1 \end{pmatrix} = \Sigma_\alpha = \Sigma_1^{\alpha}$, $p > 3$.	$\delta_{ij} = 0\ (j > i),\ \delta_{22}^2 = \delta_{11}\delta_{33},\ \delta_{33}^2 = \delta_{22}\delta_{44},$ $\delta_{43}\delta_{33} = \delta_{32}\delta_{44},\ \delta_{21}\delta_{33} = \mu\delta_{22}\delta_{32},\ \delta_{31}\delta_{44} = \mu\delta_{42}\delta_{33}$

At the end of the preceding table occur the four types of C_p, with two cases for the second and fourth types. This result is in accord with that given by the theory of canonical forms[†]; the present form of S_α and Σ_α are more convenient than the types given by the other theory.

Reference is made in the table to the following conditions:

$$\begin{aligned} \mu\delta_{12}\delta_{44} + \delta_{11}\delta_{43} = 0,\ \mu\delta_{22}\delta_{34} + \delta_{33}\delta_{21} = 0 \\ \mu^2\delta_{13}\delta_{34} - \mu\delta_{22}\delta_{44} + \mu\delta_{33}\delta_{11} - \delta_{43}\delta_{21} = 0 \end{aligned} \tag{5}$$

$$\begin{aligned} q \neq 0:\ \delta_{32} = -m\delta_{23},\ \delta_{33} = \delta_{22} + q\delta_{23},\ \delta_{22}^2 + q\delta_{22}\delta_{23} + m\delta_{23}^2 = \delta_{11}\delta_{44} \\ q = 0:\ \delta_{32} = \pm m\delta_{23},\ \delta_{33} = \mp\delta_{22},\ \delta_{22}^2 + m\delta_{23}^2 = \delta_{11}\delta_{44} \end{aligned} \tag{6}$$

$$\delta_{22}^2 - \nu_1\delta_{23}^2 = \delta_{11}\delta_{44},\ \delta_{21}\delta_{44} = \delta_{43}\delta_{22} - \nu_1\delta_{43}\delta_{23},\ \delta_{31}\delta_{44} = \delta_{42}\delta_{32} - \nu_1\delta_{43}\delta_{33}. \tag{7}$$

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[*] The conditions besides these four were initially

$$\begin{aligned} \nu\delta_{11}\delta_{44} - \nu^2\delta_{21}\delta_{34} - \nu\delta_{22}\delta_{33} + \delta_{12}\delta_{43} &= 0, \\ \nu\delta_{21}\delta_{44} - \nu\delta_{11}\delta_{34} - \delta_{12}\delta_{33} + \delta_{22}\delta_{43} &= 0, \\ \nu\delta_{21}\delta_{33} - \delta_{11}\delta_{43} - \delta_{12}\delta_{44} + \nu\delta_{22}\delta_{34} &= 0, \\ \delta_{11}\delta_{33} - \delta_{21}\delta_{43} - \delta_{22}\delta_{44} + \delta_{12}\delta_{34} &= 0. \end{aligned}$$

Since $\delta_{33}, \delta_{43}, \delta_{34}, \delta_{44}$, are not all zero, the determinant of the coefficients must vanish, viz.,

$$(\nu\delta_{11}^2 - \delta_{12}^2 - \nu^2\delta_{21}^2 + \nu\delta_{22}^2)^2 - 4\nu^2(\delta_{11}\delta_{22} - \delta_{21}\delta_{12})^2 = 0,$$
$$\{\nu(\delta_{11} - \delta_{22})^2 - (\delta_{12} - \nu\delta_{21})^2\}\{\nu(\delta_{11} + \delta_{22})^2 - (\delta_{12} + \nu\delta_{21})^2\} = 0.$$

Hence $\delta_{11} = \pm\delta_{22}$, $\delta_{12} = \pm\nu\delta_{21}$. The first and second conditions become

$$\delta_{22}(\pm\delta_{44} - \delta_{33}) + \delta_{21}(\pm\delta_{43} - \nu\delta_{34}) = 0,\ \delta_{22}(\pm\delta_{43} - \nu\delta_{34}) + \nu\delta_{21}(\pm\delta_{44} - \delta_{33}) = 0.$$

Hence $\nu(\pm\delta_{44} - \delta_{33})^2 - (\pm\delta_{43} - \nu\delta_{34})^2 = 0$, so that $\delta_{33} = \pm\delta_{44}$, $\delta_{43} = \pm\nu\delta_{34}$.
The third and fourth conditions now reduce to identities.

[†] Cf. Putnam, *Amer. Journ.* vol. 24 (1902), pp. 319-366. As the context shows, t is a not-fourth power (not merely a not-square). A slight correction is needed on pp. 358-359. If $d = 4$, S is conjugate with S^m if, and only if, m is a square, while 10 (t) and 10'' (t) generate conjugate cyclic groups; likewise, 10' (t) and 10''' (t).

5. LEMMA. *Every binary transformation of determinant unity can be generated by the* $S_\lambda = \left(\begin{smallmatrix}1 & 0\\ \lambda & 1\end{smallmatrix}\right)$ *and any* $B = \left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$, $\Delta = \alpha\delta - \beta\gamma \neq 0$, $\gamma \neq 0$.

Indeed, $S_{-\alpha\gamma^{-1}}\, B S_{-\delta\gamma^{-1}} = \begin{pmatrix} 0 & \gamma \\ -\Delta\gamma^{-1} & 0\end{pmatrix}$, which transforms S_λ into $\begin{pmatrix}1 & -\lambda\gamma^2\Delta \\ 0 & 1\end{pmatrix}$.

The subgroups H of order p^6N of G, $p > 2$.

6. In the notation of §1, we have $h = 6$, $m = 3$, 4, or 5. Let first $m = 5$. Then H has an operator S commutative with G_{p^5} but not with G_{p^6}. By §4, we may take G_{p^5} to be $\{\alpha_{s\,s-1} = 0\}$.

Let G_{p^5} be $\{\alpha_{21} = 0\}$. Then H contains G_{p^6} and an operator (δ_{ij}) with δ_{13}, δ_{14}, δ_{23}, δ_{24}, δ_{34} all zero, $\delta_{12} \neq 0$. Hence (§5), H contains

$$(8) \qquad G_{p^6(p^2-1)}: \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & 1 & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix}, \quad \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = 1.$$

The latter is a subgroup of the following groups:

$$(9) \qquad G_{p^6(p^2-1)^2(p-1)}:\ (\alpha_{ij}),\ \alpha_{13} = \alpha_{14} = \alpha_{23} = \alpha_{24} = 0, \quad |\alpha| = 1;$$

$$(10) \qquad G_{p^3(p^3-1)(p^3-p)(p^3-p^2)}:\ (\alpha_{ij}),\ \alpha_{14} = \alpha_{24} = \alpha_{34} = 0, \quad |\alpha| = 1.$$

We proceed to show that H lies in one of these groups (9), (10), so that its determination follows at once from that of the binary and ternary linear groups. We first prove that a group H, containing (8) and an operator $T = (\beta_{ij})$ in neither (9) nor (10), is identical with the total group G.

(*a*) $\beta_{14} = \beta_{24} = 0$. Then $\beta_{34} \neq 0$, β_{13} and β_{23} are not both zero. Applying on the right of T a transformation on ξ_1, ξ_2, we have $\beta_{13} \neq 0$, $\beta_{34} \neq 0$, $\beta_{14} = \beta_{24} = \beta_{23} = 0$. Applying $\xi_3' = \xi_3 + \lambda\xi_1 + \mu\xi_2$ on the left, we make also $\beta_{11} = \beta_{12} = 0$. Then applying $\xi_4' = \xi_4 + \rho\xi_1 + \sigma\xi_2$ on the left, we make also $\beta_{31} = \beta_{32} = 0$. Applying $\xi_3' = \xi_3 + \alpha\xi_1$, $\xi_4' = \xi_4 + \beta\xi_3 + \gamma\xi_1$ on the right, we make also $\beta_{33} = 0$, $\beta_{43} = \beta_{44} = 0$. Applying on the left a binary on ξ_1, ξ_2, we make $\beta_{22} = \beta_{41} = 0$, and reach

$$T_1:\ \xi_1' = \beta_{13}\xi_3,\ \xi_2' = \beta_{21}\xi_1,\ \xi_3' = \beta_{34}\xi_4,\ \xi_4' = \beta_{42}\xi_2.$$

Now (8) contains every B_{12}, B_{21}, B_{31}, B_{32}, B_{41}, B_{42}, B_{43}. But T_1 transforms B_{42} into B_{34}, B_{12} into B_{24}, B_{32} into B_{14}, B_{31} into B_{13}, B_{14} into B_{23}. Hence H contains every B_{ij}, so that $H = G$.

(*b*) β_{14}, β_{24} not both zero. Applying $\xi_1' = \xi_2$, $\xi_2' = -\xi_1$ on the right, we have $\beta_{14} \neq 0$. Applying $\xi_i' = \xi_i + \lambda_i\xi_1$ $(i = 2, 3, 4)$ on the right, we make $\beta_{14} = 0$. Applying $\xi_4' = \xi_4 + \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$ on the left, we make $\beta_{11} = \beta_{12} = \beta_{13} = 0$.

(b_1) Let $\beta_{23} \neq 0$. Applying $\xi_i' = \xi_i + \lambda_i\xi_2$ $(i = 3, 4)$ on the right, and $\xi_3' = \xi_3 + \gamma_1\xi_1 + \gamma_2\xi_2$ on the left, we make $\beta_{21} = \beta_{22} = \beta_{33} = \beta_{43} = 0$. Applying on the left a binary on ξ_1, ξ_2, we make also $\beta_{31} = \beta_{42} = 0$. There results $(\xi_1\xi_4)(\xi_2\xi_3)\cdot M$, M multiplying each ξ_i by a constant. This transforms B_{21}, B_{31}, B_{41}, B_{32}, B_{42} into B_{34}, B_{24}, B_{14}, B_{23}, B_{13}, respectively. Hence $H = G$.

(b_2) Let $\beta_{23} = 0$. Applying on the left a binary on ξ_1, ξ_2, we make $\beta_{23} = 0$. Applying on the right $B_{i2} (i = 3, 4)$, we make $\beta_{i1} = 0$. If $\beta_{33} \neq 0$, we apply B_{43} on the right and B_{32} on the left and make $\beta_{43} = \beta_{32} = 0$, and reach $(\xi_1\xi_4\xi_2)\cdot M$. If $\beta_{33} = 0$, we apply B_{32} on the left and make $\beta_{42} = 0$, reaching $(\xi_1\xi_4\xi_3\xi_2)\cdot M$. In either case, $H = G$.

Next, let H contain the subgroup (8), and an operator (β) in (9) but not in (10), and an operator (β') in (10), but not in (9). From (β) and (8) we readily obtain (§5) all binary operators on ξ_3, ξ_4. But the product of (β') by $\xi_3' = \xi_4$, $\xi_4' = -\xi_3$ gives an operator in neither (9) nor (10). Hence $H = G$.

7. Let next G_{p^5} be $\{\alpha_{43} = 0\}$. Then H contains

$$(11) \qquad G'_{p^6(p^2-1)}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & 1 & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}, \quad \alpha_{33}\alpha_{44} - \alpha_{43}\alpha_{34} = 1\,.$$

The latter lies in (9) and

$$(12) \qquad G'_{p^3(p^2-1)(p^3-p)(p^3-p^2)}: \ (\alpha_{ij}),\ \alpha_{12} = \alpha_{13} = \alpha_{14} = 0\,,\ |\alpha| = 1\,.$$

We show that H lies in (9) or (12). This follows from the result of §6 and the fact that the matrices of the general operators of (8), (9), (10) become the matrices of the general operators of (11), (9), (12), respectively, upon reflection on their left diagonals.

Indeed, every group of n-ary transformations*

$$(\alpha): \qquad \xi_i' = \sum_{j=1}^{n} \alpha_{ij}\xi_j \qquad (i = 1, \ldots, n)$$

is simply isomorphic with the group of the transformations $\{\alpha\}^{-1}$, inverse of

$$\{\alpha\}: \qquad \xi_i' = \sum_{j=1}^{n} \alpha_{n+1-j\; n+1-i}\,\xi_j \qquad (i = 1, \ldots, n),$$

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* We thus obtain without computation the result of Burnside on the ternary groups (a_{ij}), $a_{12} = a_{13} = 0$, and (a_{ij}), $a_{13} = a_{23} = 0$, *Proc. L. M. S.*, vol. 26 (1894), pp. 94-95.

where the matrix of $\{\alpha\}$ is obtained by reflecting the matrix of (α) on its left diagonal. We have only to show that

$$\text{if } (\alpha) \sim \{\alpha\}^{-1},\ (\beta) \sim \{\beta\}^{-1}, \text{ then } (\alpha)(\beta) \sim \{\alpha\}^{-1}\{\beta\}^{-1} = (\{\beta\}\{\alpha\})^{-1}.$$

This is true since

$$(\alpha)(\beta): \qquad \xi_i' = \sum_{k=1}^{n} \gamma_{ik}\xi_k,\ \gamma_{ik} \equiv \sum_{j=1}^{n} \beta_{ij}\alpha_{jk} \qquad (i, k = 1, \ldots, n),$$

$$\{\beta\}\{\alpha\}: \qquad \xi_i' = \sum_{k=1}^{n} \gamma_{ik}'\xi_k,\ \gamma_{ik}' \equiv \sum_{j=1}^{n} \beta_{n+1-k\, n+1-j}\alpha_{n+1-j\, n+1-i},$$

$$\gamma_{ik}' = \sum_{j_1=1}^{n} \beta_{n+1-k\, j_1}\alpha_{j_1\, n+1-i} = \gamma_{n+1-k\, n+1-i}.$$

8. Let finally G_{p^6} be $\{\alpha_{32} = 0\}$. Then H contains

$$G''_{p^6(p^2-1)}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 1 \end{pmatrix},\ \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} = 1. \tag{13}$$

This lies in (10) and (12). By a proof of character similar to that in §6, making the separation of cases $\beta_{14} = 0$, $\beta_{14} \neq 0$, it is found that, if H contains (13) either $H = G$, or else H lies in (10) or (12).

9. Let next $m = 4$ $(h = 6)$. By §4, we may take $G_{p^4} = \{\alpha_{21} = \alpha_{43} = 0\}$. Then H contains an operator S with $\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0$, δ_{12}, δ_{34} not both zero. Now S transforms $[\alpha_{ij}]$ into (β_{ij}), with $\beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = 0$,

$$\beta_{11} = 1 + \alpha_{21}\delta_{12}\delta_{22}\Delta_1^{-1},\ \beta_{12} = -\alpha_{21}\delta_{12}^2\Delta_1^{-1},\ \beta_{21} = \alpha_{21}\delta_{22}^2\Delta_1^{-1},$$
$$\beta_{22} = 1 - \alpha_{21}\delta_{12}\delta_{22}\Delta_1^{-1}, \qquad \Delta_1 \equiv \delta_{11}\delta_{22} - \delta_{12}\delta_{21},$$

the expressions for $\beta_{i+2\, j+2}$ being derived from those for β_{ij} $(i \overline{\gtrless} 2, j \overline{\gtrless} 2)$ by adding 2 to each subscript. Hence if $\delta_{12} = 0$, G_{p^4} and $S^{-1}G_{p^4}S$ have $\{\alpha_{43} = 0\}$ in common, contrary to $m = 4$. Hence $\delta_{12} \neq 0$, and similarly, $\delta_{34} \neq 0$. Then, by §5, H contains the group (9). But (9) is a maximal subgroup of G (§6).

10. Let finally $m = 3$ $(h = 6)$. By §1, the order of H must be a multiple of $p^6(p^4-1)(p^3-1)$. But* G has no subgroup of index $< p^3 + p^2 + p + 1$ or 27, according as $p > 3$, or $p = 3$. Similarly, the case $m = 2$, $h = 5$ is excluded, since $p(p^2 - 1) < p^3$. A partial summary of our results is given by the

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* *Transactions A. M. S.*, vol. 6 (1905), pp. 48-57.

THEOREM. *The only maximal subgroups of order a multiple of p^6 are conjugate with* (9) *or with the simply isomorphic groups* (10) *and* (12).

The subgroups H of order p^5N, N prime to p, $p > 2$.

11. Let $m = 4$ $(h = 5)$. We may set $G_{p^4} = \{\alpha_{21} = \alpha_{43} = 0\}$; otherwise, there is (§1) an operator S commutative with G_{p^4}; it has every $\delta_{ij} = 0$ $(j > i)$ and hence transforms G_{p^5} into a subgroup G'_{p^5} of G_{p^6}, where G'_{p^5} and G_{p^5} are distinct and hence generate G_{p^6}, contrary to $h = 5$. Thus H contains an operator $S = (\delta_{ij})$, with $\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0$, which is commutative with $G_{p^4} = \{\alpha_{21} = \alpha_{43} = 0\}$ and not with G_{p^5}. The latter may be taken to be $\{\alpha_{21} = 0\}$, $\{\alpha_{43} = 0\}$ or $\{\alpha_{21} = \mu\alpha_{43}\}$. In view of §7, we treat only one of the first two cases.

(*a*) Let G_{p^5} be $\{\alpha_{43} = 0\}$. Then S transforms $[\alpha]$, with $a_{43} = 0$, into $B = (\beta_{ij})$, where $\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are given in §9, while $\beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}, \beta_{43}, \beta_{34}$ are all zero, $\beta_{33} = \beta_{44} = 1$. Hence $\delta_{12} \neq 0$. Multiplying B on the left by a suitably chosen operator of G_{p^4}, the product becomes a binary transformation $\begin{pmatrix}\beta_{11}\beta_{12}\\ \beta_{21}\beta_{22}\end{pmatrix}$ on ξ_1, ξ_2, with $\beta_{12} \neq 0$. This with the $\begin{pmatrix}1 & 0\\ \alpha_{21} & 1\end{pmatrix}$ generate every binary transformation on ξ_1, ξ_2, of determinant 1 (§5). Hence H contains

(14) $\qquad G_{p^5(p^2-1)}$: operators (8) with $\beta_{43} = 0$.

Suppose that H contains an operator (β_{ij}) not in (14).

(a_1) If in every such (β_{ij}), $\beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = 0$, we apply on the right a binary on ξ_1, ξ_2 and make $\beta_{11} = 1$, $\beta_{12} = \beta_{21} = 0$, then apply $B_{41}, B_{42}, B_{31}, B_{32}$ and reach DC, where C is $\xi'_2 = \beta\xi_2$, $\xi'_3 = \beta^{-1}\xi_3$, and D is a binary on ξ_3, ξ_4 of determinant 1. The operators of H are all of the form $DC\Gamma$, Γ in (14). The operators D must form a group of order prime to p (since $h = 5$) and hence of order c, $4d$, 24, 48, 120, being cyclic, dicyclic, or the homogeneous forms of the groups of the regular bodies. The form of H is thus evident.

(a_2) If in one $B = (\beta_{ij})$, $\beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}$ are not all zero, the order of H will be shown to be a multiple of p^6, contrary to $h = 5$. We may take β_{13}, β_{14} not both zero, using BB_{12} if necessary. If $\beta_{13} = 0$, we normalize by transforming by $\xi'_3 = \xi_4$, $\xi'_4 = -\xi_3$, so that (14) is unaltered. Hence we have $\beta_{14} \neq 0$. Applying B_{i1} on the right, we make $\beta_{i4} = 0$ $(i = 2, 3, 4)$. Applying B_{41}, B_{42} on the left, we make also $\beta_{11} = \beta_{12} = 0$. If $\beta_{13} \neq 0$, we normalize by transformation by B_{43}, so that (14) is unaltered, and make $\beta_{13} = 0$. If $\beta_{23} \neq 0$, we may make $\beta_{21} = \beta_{22} = \beta_{33} = \beta_{43} = 0$. Applying on the left a binary on ξ_1, ξ_2, we reach

$(\xi_1\xi_4)(\xi_2\xi_3)\cdot M$. This transforms B_{12} into B_{43}, which extends $\{\alpha_{43}\}$ to G_{p^6}. If $\beta_{23}=0$, we may make $\beta_{22}=\beta_{31}=\beta_{41}=0$. If then $\beta_{33}\neq 0$, we may make $\beta_{32}=0$, $\beta_{42}=1$, and reach

$$\xi_1'=\beta_{14}\xi_4,\ \xi_2'=\beta_{21}\xi_1,\ \xi_3'=\beta_{33}\xi_3,\ \xi_4'=\xi_2+\beta_{43}\xi_3.$$

This transforms $\xi_3'=\xi_3+\xi_2$ into

$$\xi_1'=\xi_1,\ \xi_2'=\xi_2,\ \xi_3'=(1-\beta_{43})\,\xi_3+\beta_{33}\xi_4,\ \xi_4'=-\frac{\beta_{43}^2}{\beta_{33}}\xi_3+(1+\beta_{43})\,\xi_4.$$

This is commutative with $\{\alpha_{43}=0\}$, extenting it to G_{p^6}. Finally, if $\beta_{33}=0$, we make $\beta_{42}=0$ and reach $(\xi_1\xi_4\xi_3\xi_2)\cdot M$. Its square transforms B_{21} into B_{43}.

(b) Let G_{p^5} be $\{\alpha_{21}=\mu\alpha_{43}\}$. Then δ_{12}, δ_{34} are not both zero in S.

(b_1) Let first $\delta_{34}=0$, $\delta_{12}\neq 0$. We transform $[\alpha]$, with $\alpha_{21}=\mu\alpha_{43}\neq 0$, by S and apply on the left a suitably chosen operator of G_{p^5} and obtain a binary $B=\begin{pmatrix}\alpha&\gamma\\ \beta&\delta\end{pmatrix}$ on ξ_1, ξ_2 of determinant 1 and having $\gamma\neq 0$. Now G_{p^5} contains $B_{21\lambda}B_{43\lambda\mu^{-1}}\equiv\Sigma_\lambda$. As in §5,

$$\Sigma_{-\alpha\gamma^{-1}}\,B\Sigma_{-\delta\gamma^{-1}}=\begin{pmatrix}0&\gamma\\ -\gamma^{-1}&0\end{pmatrix}_{1,2}B_{43\rho},\ \rho\equiv-(\alpha+\delta)\gamma^{-1}\mu^{-1}.$$

This transforms Σ_λ into $\begin{pmatrix}1&-\lambda\gamma^2\\ 0&1\end{pmatrix}_{1,2}B_{4,3,\lambda\mu^{-1}}\equiv U_{\lambda,\gamma}$. Then

$$U_{\lambda,\gamma}\Sigma_\lambda^{-1}=\begin{pmatrix}1&-\lambda\gamma^2\\ -\lambda&1+\lambda^2\gamma^2\end{pmatrix}_{1,2}.$$

Hence H contains a binary transformation B in which γ is arbitrary, and hence also $U_{1,\frac{1}{2}(r-1)}\,U^{-1}_{1,\frac{1}{2}(r+1)}\equiv B_{1,2,r}$, r arbitrary. Then from $U_{\lambda,\gamma}$, we obtain every B_{43} in H, and hence also B_{21}. Hence H contains G_{p^5}.

(b_2) The case $\delta_{12}=0$, $\delta_{34}\neq 0$ is excluded similarly, or by §7.

(b_3) Let finally $\delta_{12}\neq 0$, $\delta_{34}\neq 0$. As in (b_1), we readily obtain from S a transformation B_1B_2, where $B_1=\begin{pmatrix}\beta_{11}&\beta_{12}\\ \beta_{21}&\beta_{22}\end{pmatrix}_{1,2}$, of determinant 1, having $\beta_{12}\neq 0$, and $B_2=\begin{pmatrix}\beta_{33}&\beta_{34}\\ \beta_{43}&\beta_{44}\end{pmatrix}_{3,4}$, of determinant 1, having $\beta_{34}\neq 0$. We proceed on the latter as in (b_1) and reach $C=C_1C_2$, where $C_1=\begin{pmatrix}\alpha&\gamma\\ \beta&\delta\end{pmatrix}$, $\gamma\neq 0$, $C_2=\begin{pmatrix}1&1\\ 0&1\end{pmatrix}$. We transform by $\xi_2'=\xi_2+\rho\xi_1$, which transforms G_{p^4} and G_{p^5} into themselves, and make $\alpha=\gamma\mu$. Then

$$T\equiv\Sigma_{-\mu}C\Sigma_{-\delta\gamma^{-1}}=\begin{pmatrix}0&\gamma\\ -\gamma^{-1}&0\end{pmatrix}_{1,2}\begin{pmatrix}0&1\\ -1&b\end{pmatrix}_{3,4},\ b=1-\frac{\delta}{\mu\gamma}.$$

Then $T^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_{1,2} \begin{pmatrix} -1 & b \\ -b & b^2-1 \end{pmatrix}_{3,4}$. Hence $b = 0$ by (b_2). Then T transforms $\Sigma_{-\mu}$ into $B_{1,2,\mu\gamma^2} B_{3,4,1} \equiv W$. Then $W^{-1} B_{2,1,\mu} B_{4,3,1} WT$ is of the form $B_1 B_2$, with $\beta_{12} = \gamma(1-\mu^2\gamma^2)$, $\beta_{34} = 0$. Hence $\mu^2\gamma^2 = 1$ by (b_1). The group generated by Σ_λ and W is composed of the operators

$$(15) \qquad \begin{pmatrix} \alpha & \gamma\mu^{-1} \\ \beta\mu & \delta \end{pmatrix}_{1,2} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}_{3,4}, \quad \alpha\delta - \beta\gamma = 1.$$

These with $\{\alpha_{21} = \alpha_{43} = 0\}$ generate a $G'_{p^5(p^2-1)}$.

Let H have a further operator $B = (\beta_{ij})$. If $\beta_{13} = \beta_{14} = \beta_{23} = \beta_{24} = 0$, we reach $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{1,2} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}_{3,}$. Then $b = 0$ by (b_1). Hence H contains $Y = \begin{pmatrix} a & 0 \\ c & a^{-1}e^{-1} \end{pmatrix}_{1,2} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}_{3,4}$. Let first $c \neq 0$. Then $B_{12,\lambda\mu^{-1}} B_{3,4,\lambda}$ transforms Y into $\begin{pmatrix} \cdot & r \\ c & \cdot \end{pmatrix}_{1,2} \begin{pmatrix} e & \lambda - \lambda e \\ 0 & 1 \end{pmatrix}_{3,4}$, $r = \frac{\lambda}{\mu}(a^{-1}e^{-1} - a - \frac{\lambda}{\mu}c)$. If $e \neq a^{-2}$, we can take $\lambda \neq 0$ to make $r = 0$. Then $e = 1$ by (b_2), and

$$\Sigma_{-\lambda\mu} Y^{-1} \Sigma_{\lambda\mu} Y = \begin{pmatrix} 1 & 0 \\ \lambda\mu(a^{-2}-1) & 1 \end{pmatrix}_{1,2},$$

whence $a^{-2} = 1 = e$ by (b_1). There remains the case $e = a^{-2}$. Then

$$Y^{-1} \Sigma_{\lambda\mu}^{-1} Y \Sigma_{\lambda\mu} = \begin{pmatrix} 1 & 0 \\ \lambda(1-a^2) & 1 \end{pmatrix}_{3,4},$$

whence $a^2 = 1$. Then $Y^2 = B_{2,1,\pm 2c}$, giving a G_{p^6}. Hence must $c = 0$ in Y. Then

$$\Sigma_{-e^{-1}\mu} Y^{-1} \Sigma_\mu Y = B_{2,1,\rho}, \quad \rho = \mu e^{-1}(a^{-2}-1),$$

whence $a^2 = 1$. Then $Y = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm e^{-1} \end{pmatrix}_{1,2} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}_{3,4}$ transforms $G'_{p^5(p^2-1)}$ into itself and extends it to a group of order $p^5(p^2-1)t$, t a factor of $2(p-1)$. Replacing α, γ by $e^{-1}\alpha, e^{-1}\gamma$, in the product of (15) by Y, we obtain (19).

Suppose next that $\beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}$ are not all zero in B. We readily make $\beta_{14} = 1$, $\beta_{11}, \beta_{12}, \beta_{13}, \beta_{24}, \beta_{34}, \beta_{44}$ all zero,

Let $\beta_{23} \neq 0$. We readily make $\beta_{21} = \beta_{22} = \beta_{33} = \beta_{43} = 0$. If $\beta_{32} = 0$, $\beta_{41} = 0$, then $B = (\xi_1\xi_4\xi_2\xi_3) M$ and $H = G$. If $\beta_{32} = 0$, $\beta_{41} \neq 0$, then $BT_\lambda BT_\rho$, where $T_\lambda \equiv B_{1,2,\lambda\mu^{-1}} B_{3,4,\lambda}$, has the form

$$(16) \qquad \begin{pmatrix} g & h \\ j & k \end{pmatrix}_{1,2} \begin{pmatrix} u & v \\ w & z \end{pmatrix}_{3,4},$$

$h=\beta_{42}(1+\mu^{-1}\rho\lambda\beta_{23})$, $v=\beta_{31}+\rho\beta_{41}$. We may choose ρ and λ to make $v=0$, $h\neq 0$, contrary to (b_1). Hence $\beta_{32}\neq 0$. Then $(B\Sigma_{\lambda\mu})^2$ is of the form (16) with $h=\beta_{43}+\lambda\beta_{32}$, $v=\beta_{31}+\lambda\mu\beta_{32}$; hence $\beta_{31}=\mu\beta_{43}$ by (b_2). If $\beta_{43}=0$, $B=(\xi_1\xi_4)(\xi_2\xi_3)\cdot M$, and $H=G$. Hence B has the form

$$\xi_1'=\xi_4,\ \xi_2'=c\xi_3,\ \xi_3'=\mu d\xi_1+b\xi_2,\ \xi_4'=e\xi_1+d\xi_2,\ cbd\neq 0. \tag{17}$$

Then $\{(15)B\}^2$ is of the form (16) with

$$h=\beta(d\gamma+b\delta)+\mu^{-1}e\gamma\delta+d\delta^2,\ v=\mu da\delta+\mu b\beta\delta+bc\gamma\delta+cd\gamma^2.$$

If $\delta\neq 0$, we can choose α to satisfy $\alpha\delta-\beta\gamma=1$ and $v=0$ if

$$\mu d+(\mu\beta+c\gamma)(d\gamma+b\delta)=0. \tag{18}$$

We can give to $d\gamma+b\delta$ any value $t\neq 0$ by choice of γ and then satisfy (18) by choice of β. The resulting value of h is

$$-d+\frac{\delta t}{d\mu}(e+bc)-\frac{ct^2}{d\mu}+\delta^2\left(d-\frac{be}{d\mu}\right).$$

By (b_1) this must be zero for every $t\neq 0$, $\delta\neq 0$. Hence $p=3$, $e=-bc$. The determinant of (17) is $-c(\mu d^2-be)=1$. Hence $p=3$, $c=\mu$, $e=-\mu b$. Then the square of (17) is

$$L\equiv\begin{pmatrix}-\mu b & d\\ d & \mu b\end{pmatrix}_{1,2}\begin{pmatrix}\mu b & \mu d\\ \mu d & -\mu b\end{pmatrix}_{3,4}$$

The product of (15) with $\alpha=\mu b$, $\gamma=\mu d$, $\beta=\mu d$, $\delta=-\mu b$, by L^{-1} is $V^{\pm 1}$,

$$V=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}_{1,2};\ \Sigma_1^{-1}V^{-1}\Sigma_1V^{-1}=\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}_{1,2},$$

contrary to (b_1).

12. Let $m=3$ $(h=5)$. The 5 possible G_{p^3} are studied under (a)—(e).

(a) Let G be $\{\alpha_{21}=\alpha_{32}=\alpha_{31}=0\}$. Then H has an operator $S=(\delta_{ij})$ with $\delta_{14}=\delta_{24}=\delta_{34}=0$, δ_{12}, δ_{13}, δ_{23} not all zero. As the G_{p^5} we may take $\{\alpha_{21}=0\}$, $\{\alpha_{32}=0\}$, or $\{\alpha_{21}=\alpha_{32}\}$.

(a_1) Let G_{p^5} be $\{\alpha_{21}=0\}$. Then δ_{13}, δ_{23} are not both zero; otherwise S transforms G_{p^5} into itself (§9, or §4). We may normalize by transforming by $\xi_1'=\xi_2$, $\xi_2'=-\xi_1$, and set $\delta_{13}\neq 0$. Applying operators of G_{p^5}, we may make δ_{11}, δ_{12}, δ_{33}, δ_{41}, δ_{42}, δ_{43} all zero. Normalizing by transformation by B_{21}, we make also $\delta_{23}=0$. If $\delta_{23}\neq 0$, SB_{32} has $\delta_{32}=0$, and transforms B_{31} into B_{13}. The commutator of B_{13}, B_{32} is B_{12}. But G_{p^5} and B_{12} generate a G'_{p^6}, the transform of G_{p^5} by $(\xi_1\xi_2)$. If $\delta_{22}=0$, SB_{32} has $\delta_{31}=0$, giving $(\xi_1\xi_3\xi_2)\cdot M$. Its square transforms B_{32} into B_{21}, giving G_{p^5}.

(a_2) Let G_{p^5} be $\{\alpha_{32}=0\}$. Then δ_{12}, δ_{13} are not both zero (§4). Normalizing by transformation by $\xi'_2=\xi_3$, $\xi'_3=-\xi_2$, we may set $\delta_{13}\neq 0$. We readily make $\delta_{11}=\delta_{23}=\delta_{33}=\delta_{41}=\delta_{42}=\delta_{43}=0$. Normalizing by B_{32}, we make $\delta\ \ =0$. If $\delta_{22}=0$, $B_{21}S$ has $\delta_{31}=0$, giving $(\xi_1\xi_3\xi_2)\cdot M$. This transforms B_{21} into B_{32}, giving G_{p^5}. If $\delta_{22}\neq 0$, $B_{21}S$ has $\delta_{21}=0$, and transforms $B_{3,1,1}$ into

$$\xi'_2=(1-\delta_{32}\delta_{31}^{-1})\,\xi_2+\delta_{22}\delta_{31}^{-1}\xi_3,\ \xi'_3=-\delta_{32}^2\delta_{31}^{-1}\delta_{22}^{-1}\xi_2+(1+\delta_{32}\delta_{31}^{-1})\,\xi_3,$$

which is commutative with $\{\alpha_{32}=0\}$ by §4 and extends it to a G'_{p^5}.

(a_3) Let G_{p^5} be $\{\alpha_{21}=\alpha_{32}\}$. The operators in H with $\delta_{14}=\delta_{24}=\delta_{34}=0$ form a subgroup H_1 containing G_{p^5}. The partial transformations on ξ_1, ξ_2, ξ_3 form a group H' of order p^3l, l prime to p. It contains G_{p^2}: $\xi'_2=\xi_2+\alpha\xi_1$, $\xi'_3=\xi_3+\alpha\xi_2+\beta\xi_1$. The group H'_1 of the operators of determinant 1 of H' contain G_{p^2} self-conjugately.* This is also true of H' since the number of conjugates to G_{p^2} in H' is a divisor of $p-1$. Hence $\delta_{12}=\delta_{13}=\delta_{23}=0$ in every operator (δ_{ij}) of H', contrary to (a).

(b) Let G_{p^3} be $\{\alpha_{32}=\alpha_{43}=\alpha_{42}=0\}$. This case is reduced to case (a) by means of the general method of §7.

(c) Let G_{p^3} be $\{\alpha_{21}=\alpha_{43}=0,\alpha_{41}=\mu\alpha_{32}\}$. Then H has an operator $S=(\delta_{ij})$, with $\delta_{13}=\delta_{14}=\delta_{23}=\delta_{24}=0$. Now G_{p^5} may be taken to be $\{\alpha_{21}=0\}$, $\{\alpha_{43}=0\}$, or $\{\alpha_{21}=\mu\alpha_{43}\}$. But in each case, G_{p^5} and $S^{-1}G_{p^5}S$ have $\{\alpha_{21}=\alpha_{43}=0\}$ in common (§9), contrary to $m=3$.

(d) Let G_{p^3} be $\{\alpha_{32}=0,\ \alpha_{21}=\mu\alpha_{43},\ \alpha_{31}=-\mu\alpha_{42}\}$. Then H has an operator $S=(\delta_{ij})$, with $\delta_{12}=\delta_{13}=\delta_{14}=\delta_{24}=\delta_{34}=0$, $\delta_{23}\neq 0$, $\delta_{22}\delta_{33}-\delta_{23}\delta_{32}=\delta_{11}\delta_{44}=\pm 1$. Since $\{\alpha_{32}=0\}$ is excluded (§4), G_{p^5} may be taken to be $\{\alpha_{21}=\mu\alpha_{43}\}$ or $\{\alpha_{21}=\alpha_{32}+\mu\alpha_{43}\}$. In either case we apply on the right and left of S operators of G_{p^3} and get

$$D:\ \xi'_1=\delta_{11}\xi_1,\ \xi'_2=\delta_{23}\xi_3,\ \xi'_3=\delta_{31}\xi_1\mp\delta_{23}^{-1}\xi_2,\ \xi'_4=\pm\delta_{11}^{-1}\xi_4.$$

This transforms B_{421} into $\xi'_4=\xi_4-\delta_{23}\delta_{11}^{-1}\xi_3+\delta_{23}\delta_{31}\delta_{11}^{-2}\xi_1$. Hence H contains B_{43} and hence also G_{p^5}.

(e) Let G_{p^3} be $\{\alpha_{32}=0,\alpha_{42}=\alpha_{21},\alpha_{31}=q\alpha_{21}+m\alpha_{43}\}$. Then G_{p^5} is $\{\alpha_{32}=0\}$. But every operator commutative with G_{p^3} is commutative with G_{p^5} (§4).

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* *American Journ. Math.*, vol. 26 (1905), p. 198, §7.

THEOREM. *A subgroup of order p^5N, N prime to p, $p > 2$, either has an invariant G_{p^5}, when its form is readily determined from* §4, *or is conjugate with one of the following types of groups*: (9) *with the* $\begin{pmatrix} \alpha_{11}\alpha_{12} \\ \alpha_{21}\alpha_{22} \end{pmatrix}$ *defining a binary group of order prime to p*; (9) *with the same restriction on the* $\begin{pmatrix} \alpha_{33}\alpha_{34} \\ \alpha_{43}\alpha_{44} \end{pmatrix}$; *a group of $p^5(p^3-1)t$ operators, t a factor of* $2\,(p-1)$:

$$(19) \qquad \begin{pmatrix} \pm\,\alpha e^{-1} & \pm\,\mu^{-1}\gamma e^{-1} & 0 & 0 \\ \pm\,\mu\beta e^{-1} & \pm\,\delta e^{-1} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha & \gamma \\ \alpha_{41} & \alpha_{42} & \beta & \delta \end{pmatrix}, \quad \begin{matrix} \alpha\delta - \beta\gamma = e, \\ \mu \text{ as in } \S 4. \end{matrix}$$

13. The results in the earlier sections suffice for a similar treatment of the subgroups of order p^4N; also for those of order p^3N, special considerations being necessary for the case in which no two G_{p^3} have a common operator other than identity, N being a multiple of $(p^4-1)(p^3-1)$.

THE UNIVERSITY OF CHICAGO, *November*, 1904.

On the Integration of a System of Differential Equations in Kinematics.

By John Eiesland.

If we have a rigid system moving about a fixed point 0 and p, q, r be the components of rotation, in order to determine the motion completely we must integrate the following system of differential equations,

$$(1) \qquad \begin{cases} \dfrac{d\alpha}{dt} = \beta r - \gamma q, \\ \dfrac{d\beta}{dt} = \gamma p - \alpha r, \\ \dfrac{d\gamma}{dt} = \alpha q - \beta p, \end{cases}$$

which has been treated by Euler, D'Alembert, and more recently by Darboux (Theorie des Surfaces, vol. I, Chap. 1–2). Euler expresses the 9 cosines of the variable axes of the system in terms of three angles θ, ψ, ϕ (the so-called Euler's angles) which, if known, will determine the motion of the system.

Darboux reduces the problem of integration to that of a Riccati equation by introducing two new variables x and y which remain constant respectively on each set of rectilinear generators of the sphere $\alpha^2 + \beta^2 + \gamma^2 = 1$ (Leçons, vol. I, p. 22). This method, while very elegant, does not admit of any obvious extension to systems with four and in general n variables such as have been deduced by Professor Craig and Mr. Hatzidakis,* while the introduction of angles corresponding to Euler's angles becomes impracticable for higher dimensions.

In a note published in Am. Jour. vol. 20, I have sketched an integration theory for a system in 4-dimensional space adopting a method employed by Lie to the general linear system in two and three variables. The essential feature of this method is a transformation of the co-ordinates by means of the formulæ

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* Am. Jour. of Math., vols. 20 and 23.

$$\alpha_1 = \frac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2 + \dots + \rho_{n-1}^2 + 1}}, \quad \alpha_2 = \frac{\rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + \dots + \rho_{n-1}^2 + 1}}, \dots\dots,$$

$$\alpha_n = \frac{1}{\sqrt{\rho_1^2 + \rho_2^2 + \dots + \rho_{n-1}^2 + 1}},$$

which is nothing but an introduction of homogeneous variables. The homogeneous system is thereby transformed into a system of generalized Riccati equations in $n - 1$ variables whose coefficients are functions of t as before.

A remark may here be made concerning this transformation. Consider the case of three variables

$$\alpha_1 = \frac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \alpha_2 = \frac{\rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \alpha_3 = \frac{1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}.$$

If α_1, α_2, α_3 be a point on the unit sphere, then, by central projection of the point on the plane $\alpha_3 = -1$, the corresponding point in this plane will have the co-ordinates ρ_1 and ρ_2, the point of intersection of the α_3 axis with this plane being taken as the origin. To a rotation of the sphere there corresponds a certain motion of the points on the plane, namely a Non-Euclidean motion whose absolute is the circle $\rho_1^2 + \rho_2^2 + 1 = 0$.*

The transformed system

$$\frac{d\rho_1}{dt} = -q + \rho_2 r + \rho_1 (\rho_2 p - \rho_1 q),$$

$$\frac{d\rho_2}{dt} = p - \rho_1 r + \rho_2 (\rho_2 p - \rho_1 q),$$

is, however, not easier to integrate than the original system. The only advantage we can draw from this transformation is to render the problem more intuitive, if we intepret ρ_1, ρ_2, t as coordinates of three-dimensional space, (See Lie-Scheffer, Cont. Gr. p. 778), which advantage is lost in the extension of the problem to four or more variables. Moreover, the Euclidean Motion on the sphere is more intuitive for very good reasons than the motion of the Non-Euclidean manifoldness on which the sphere may be projected centrally. We shall therefore not use this transformation, but consider the system in its homogeneous form.

It is the object of this paper to show how by another method that is capable of generalization the integration problem may be reduced to its simplest terms.

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* See Klein, Nicht-Euclidische Geometrie, Vorlesungshefte, I, p. 228.

We shall use geometrical language as much as possible and integrate geometrically in the sense employed by Lie in his Berührungstransformationen, and in Lie-Scheffer's volume on continuous groups. The analytical treatment may then be considered as a verification of theorems that have been made evident from geometrical considerations.

I.

We shall begin with the case of two variables and consider the equations

$$\left\{\begin{aligned} \frac{dx_1}{dt} &= \quad p_{12}x_2, \\ \frac{dx_2}{dt} &= -p_{12}x_1, \end{aligned}\right. \tag{2}$$

where p_{12} is a given function of t. These equations admit of the Euclidean group of rotations around the origin of the plane x_1x_2, that is, the transformation

$$Uf = -p_{12}\frac{\partial f}{\partial x_1} + p_{12}\frac{\partial f}{\partial x_2};$$

but since t is a variable we must conceive of the transformation as continually changing with the time t. Now since all the circles $x_1^2 + x_2^2 =$ const. are invariant curves of the system, the two lines

$$x_1 = ix_2, \quad x_1 = -ix_2,$$

which pass through the circular points at infinity are common tangents to all these circles and may therefore be considered as their envelope; hence they are a pair of integral curves of (2). But any system of the form

$$\frac{dx_1}{dt} = \alpha_1 x_1 + \beta_1 x_2,$$

$$\frac{dx_2}{dt} = \alpha_2 x_1 + \beta_2 x_2$$

can be solved by one quadrature if two integral curves are known, hence (2) can be solved by one quadrature only.

This may easily be verified analytically. Introducing the variable $\frac{x_1}{x_2} = \omega$ in (2) the system reduces to the form

$$\frac{d\omega}{dt} = p_{12}(1 + \omega^2)$$

which is a Riccati equation having $\omega = \pm i$ for particular solutions and can therefore be solved by a single quadrature; the general integral is

$$x_1 = \varkappa \sin \alpha, \quad x_2 = \varkappa \cos \alpha,$$

where

$$\alpha = \int p_{12}\, dt.$$

We shall now consider the system in three variables

$$(3) \qquad \begin{cases} \dfrac{dx_1}{dt} = \phantom{-p_{12}x_1} p_{12}x_2 - p_{13}x_3, \\ \dfrac{dx_2}{dt} = -p_{12}x_1 \phantom{p_{12}x_2} + p_{23}x_3, \\ \dfrac{dx_3}{dt} = p_{13}x_1 - p_{23}x_2, \end{cases}$$

which is identical with (1), if we put $p_{13} = q$, $p_{12} = r$, $p_{23} = p$. Since the ∞^1 spheres $x_1^2 + x_2^2 + x_3^2 =$ const. are invariant surfaces, if we determine the motion of any one of these, say the unit sphere, the complete motion is known; we may therefore limit ourselves to the consideration of this sphere.

Let it be supposed that we know an integral curve C on the sphere; we shall prove that the integration of (3) may be reduced to that of the system

$$(4) \qquad \frac{dy_1}{dt} = P_{12}y_2, \quad \frac{dy_2}{dt} = -P_{12}y_1, \quad \frac{dy_3}{dt} = 0.$$

Let the initial position of the x_1, x_2, x_3-axes be such that the x_3-axis passes through a point P_0 on C at the time t_0. We now transform these axes to a set of new but variable axes y_1, y_2, y_3 such that the y_3-axis describes the curve C; this transformation will be a Euclidean one whose 9 direction-cosines are functions of t depending on the given integral curve. We now pass a plane, $y_3 = 0$, through the origin and perpendicular to the y_3-axis and choose arbitrarily in the intersection of this plane with the sphere two points Q and R at a quadrant's distance from each other. As y_3 moves along C, the two axes $OQ = y_1$, $OR = y_2$ will describe a unit circle in the plane $y_3 = 0$. But since the transformation is Euclidean, the transformed system will have the same form in the variables y_1, y_2, y_3 as the original system (3). Moreover, $y_1 = y_2 = 0$ will be a particular solution of the system, that is, the system takes the form (4), *q. e. d.*

To verify this theorem by analysis we proceed as follows:

Let the integral curve be given in the form

$$(5) \qquad x_1 = \rho_1(t)\, x_3, \quad x_2 = \rho_2(t)\, x_3.$$

We transform the axis by means of the orthogonal transformation

$$(6)\qquad \begin{cases} y_1 = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3, \\ y_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3, \\ y_3 = \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3, \end{cases}$$

where the α's are the 9 direction-cosines of the new axis. Now since the y_3-axis always passes through a point P on the integral curve, $y_1 = y_2 = 0$, whenever $x_1 = \rho_1 x_3$ and $x_2 = \rho_2 x_3$; these conditions must then be written

$$\rho_1\alpha_{11} + \rho_2\alpha_{12} + \alpha_{13} \equiv 0,$$
$$\rho_1\alpha_{21} + \rho_2\alpha_{22} + \alpha_{23} \equiv 0,$$

or, more simply,

$$(7)\qquad \rho_1 = \frac{\alpha_{31}}{\alpha_{33}}, \quad \rho_2 = \frac{\alpha_{32}}{\alpha_{33}},$$

that is to say, α_{31}, α_{32}, α_{33} are known functions of t, viz.:

$$(8)\qquad \alpha_{31} = \frac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \alpha_{32} = \frac{\rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \alpha_{33} = \frac{1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}.$$

Introducing now the new coordinates in (3), we find after easy reductions, keeping always in mind the 6 relations between the 9 cosines α_{ik}

$$\begin{aligned} \frac{dy_1}{dt} = {} & [p_{12}\alpha_{33} - p_{13}\alpha_{32} + p_{23}\alpha_{31} + \alpha_{21}\alpha'_{11} + \alpha_{22}\alpha'_{12} + \alpha_{23}\alpha'_{31}]\, y_2 \\ & + [p_{12}\alpha_{23} - p_{13}\alpha_{22} + p_{23}\alpha_{21} - \alpha'_{31}\alpha_{11} - \alpha'_{32}\alpha_{12} - \alpha'_{33}\alpha_{13}]\, y_3, \\ \frac{dy_2}{dt} = {} & - [p_{12}\alpha_{33} - p_{13}\alpha_{32} + p_{23}\alpha_{31} - \alpha_{21}\alpha'_{11} - \alpha_{22}\alpha'_{12} - \alpha_{23}\alpha'_{13}]\, y_1 \\ & + [p_{12}\alpha_{13} - p_{13}\alpha_{12} + p_{23}\alpha_{11} - \alpha'_{31}\alpha_{21} - \alpha'_{32}\alpha_{22} - \alpha'_{33}\alpha_{23}]\, y_3, \\ \frac{dy_3}{dt} = {} & - [p_{12}\alpha_{23} - p_{13}\alpha_{22} + p_{23}\alpha_{21} - \alpha'_{31}\alpha_{11} - \alpha'_{32}\alpha_{12} - \alpha'_{33}\alpha_{13}]\, y_1 \\ & - [p_{12}\alpha_{13} - p_{13}\alpha_{12} + p_{23}\alpha_{11} - \alpha'_{31}\alpha_{21} - \alpha'_{32}\alpha_{22} - \alpha'_{33}\alpha_{23}]\, y_2, \end{aligned}$$

where the α'_{ik} are derivatives of the α's with respect to t. This system may be written

$$(9)\qquad \begin{cases} \dfrac{dy_1}{dt} = P_{12}y_2 - P_{13}y_3, \\ \dfrac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3, \\ \dfrac{dy_3}{dt} = P_{13}y_1 - P_{23}y_2, \end{cases}$$

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which is of the same form as the original system. But, $P_{13} = P_{23} \equiv 0$; in fact, since by (5) and (7) we have

$$(10) \qquad \frac{x_1}{x_3} = \frac{\alpha_{31}}{\alpha_{33}} = \rho_1, \quad \frac{x_2}{x_3} = \frac{\alpha_{32}}{\alpha_{33}} = \rho_2,$$

it follows that $x_1 = \alpha_{31}$, $x_2 = \alpha_{32}$, $x_3 = \alpha_{33}$ are a set of particular solutions of (3) and we must therefore also have

$$\begin{aligned} \alpha'_{31} &= \quad p_{12}\alpha_{32} - p_{13}\alpha_{33}, \\ \alpha'_{32} &= -p_{12}\alpha_{31} + p_{23}\alpha_{33}, \\ \alpha'_{33} &= \quad p_{13}\alpha_{31} - p_{23}\alpha_{32}. \end{aligned}$$

Substituting these values of α'_{31}, α'_{32} and α'_{33} in P_{13} and P_{23}, we find that they vanish identically, and our system reduces to the form

$$(4) \qquad \frac{dy_1}{dt} = P_{12}y_2, \quad \frac{dy_2}{dt} = -P_{12}y_1, \quad \frac{dy^3}{dt} = 0, \; q.e.d.$$

If now we determine the six cosines α_{11}, α_{12}, α_{13}; α_{21}, α_{22}, α_{23} in terms of ρ_1 and ρ_2, we know the function P_{12}, so that this last system may be integrated by one quadrature. Having found y_1, y_2 and y_3 we substitute in (6) and solve for x_1, x_2 and x_3 which will be the general integrals of (3). Since the knowledge of an integral curve C on the unit-sphere by (10) amounts to having a set of particular solutions given, we may state the result thus:

If a set of particular solutions of (3) *is known, the system may be integrated by one quadrature.* This is the form in which Darboux states this theorem (Leçons, vol. I, p. 28).

It only remains to show how to calculate the six cosines α_{1i}, α_{2i} $(i = 1, 2, 3)$ in terms of ρ_1 and ρ_2, or, what is the same thing, in terms of α_{31}, α_{32}, α_{33}, which are known functions of t.

The integration problem such as Euler conceived it consists in determining the 9 cosines of the new or variable axes (which are different from the axes y_1, y_2, y_3 chosen above) as functions of three angles ϕ, θ, ψ (see Darboux, vol. I, p. 3). The formulæ thus obtained being rather unsymmetrical, he adopted another system of parameters, introducing the direction-cosines α, β, ν of an axis of rotation a and a rotation ω about the axis. This representation is simplified if we introduce homogeneous parameters putting

$$A = \sin\frac{\omega}{2}\cos\alpha, \quad B = \sin\frac{\omega}{2}\cos\beta, \quad C = \sin\frac{\omega}{2}\cos\gamma, \quad D = \cos\frac{\omega}{2},$$

so that $A^2 + B^2 + C^2 + D^2 = 1$. The nine cosines are then as follows:*

$$\begin{array}{lll}
D^2 + A^2 - B^2 - C^2, & 2\,(AB - CD) \quad , & 2\,(AC - BD), \\
2\,(AB + CD) \quad , & D^2 - A^2 + B^2 - C^2, & 2\,(BC - AD), \\
2\,(AC - BD) \quad , & 2\,(BC + AD) \quad , & D^2 - A^2 - B^2 + C^2.
\end{array}$$

If we put $D = \frac{1}{\Delta}$, $A = \frac{\lambda}{\Delta}$, $B = \frac{\mu}{\Delta}$, $C = \frac{-\nu}{\Delta}$ where $\Delta = 1 + \lambda^2 + \mu^2 + \nu^2$ we obtain the parametric representation of Cayley for $n = 3$.† The table of cosines is then

$$\begin{array}{lll}
\frac{1 + \lambda^2 - \mu^2 - \nu^2}{\Delta}, & 2\frac{\nu + \lambda\mu}{\Delta} \quad , & 2\frac{\lambda\nu - \mu}{\Delta} \quad , \\
\frac{2 - \nu + \lambda\mu}{\Delta} \quad , & \frac{1 + \mu^2 - \nu^2 - \lambda^2}{\Delta}, & 2\left(\frac{\lambda + \mu\nu}{\Delta}\right) \quad , \\
2\frac{\mu + \lambda\nu}{\Delta} \quad , & 2\frac{\nu\mu - \lambda}{\Delta} \quad , & \frac{1 + \nu^2 - \lambda^2 - \mu^2}{\Delta}.
\end{array}$$

Now since these cosines are connected by two relations only, viz.:

$$\frac{\alpha_{31}}{\alpha_{33}} = \rho_1, \quad \frac{\alpha_{32}}{\alpha_{33}} = \rho_2,$$

one of these three parameters is perfectly arbitrary and may be put equal to zero; we have then the following table of cosines, putting $\lambda = 0$,

$$\begin{array}{lll}
\frac{1 - \mu^2 - \nu^2}{\Delta}, & \frac{2\nu}{\Delta}, & \frac{-2\mu}{\Delta} \quad , \\
\frac{-2\nu}{\Delta} & \frac{1 + \mu^2 - \nu^2}{\Delta}, & \frac{2\mu\nu}{\Delta} \quad , \\
\frac{2\mu}{\Delta} \quad , & \frac{2\mu\nu}{\Delta} \quad , & \frac{1 + \nu^2 - \mu^2}{\Delta},
\end{array}$$

where $\Delta = 1 + \mu^2 + \nu^2$. We have now by (7) and (8)

$$\frac{2\mu}{\Delta} = \alpha_{31} = \frac{\rho_1}{\sqrt{1 + \rho_1^2 + \rho_2^2}}, \quad \frac{2\mu\nu}{\Delta} = \alpha_{32} = \frac{\rho_2}{\sqrt{1 + \rho_1^2 + \rho_2^2}}, \quad \frac{1 + \nu^2 - \mu^2}{\Delta}$$

$$= \alpha_{33} = \frac{1}{\sqrt{1 + \rho_1^2 + \rho_2^2}}.$$

Solving these equations for μ and ν we get

$$\nu = \frac{\rho_2}{\rho_1}, \quad \mu = \frac{1 - \sqrt{1 + \rho_1^2 + \rho_2^2}}{\rho_1}.$$

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*See Encyclopaedie der Math. Wiss., Bd. IV, p. 204.

†Pascal, Die Determinanten p. 162.

Remark. This choice of λ amounts geometrically to choosing the position of the moving axes y_1 and y_2 in the plane $y_3 = 0$ in such a way that the direction-cosines of y_2 become $\alpha_{21}, \alpha_{22}, \alpha_{32}$, and those of y_1, $\alpha_{11}, -\alpha_{21}, -\alpha_{31}$.

Expressing the above cosines in terms of ρ_1 and ρ_2 we obtain the table

$$\begin{array}{ccc}
\dfrac{\rho_1^2-\rho_2^2\sqrt{1+\rho_1^2+\rho_2^2}}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{\rho_1\rho_2\left(1+\sqrt{1+\rho_1^2+\rho_2^2}\right)}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{-\rho_1}{\sqrt{1+\rho_1^2+\rho_2^2}}, \\
\dfrac{-\rho_1\rho_2\left(1+\sqrt{1+\rho_1^2+\rho_2^2}\right)}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{-\rho_2^2+\rho_1^2\sqrt{1+\rho_1^2+\rho_2^2}}{(\rho_1^2+\rho_2^2)\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{+\rho_2}{\sqrt{1+\rho_1^2+\rho_2^2}}, \\
\dfrac{\rho_1}{\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{\rho_2}{\sqrt{1+\rho_1^2+\rho_2^2}}, & \dfrac{1}{\sqrt{1+\rho_1^2+\rho_2^2}},
\end{array}$$

or, in terms of α_{31}, α_{32} and α_{33} which is a set of known particular solutions,

$$\begin{array}{ccc}
\dfrac{\alpha_{33}+\alpha_{31}^2-1}{1-\alpha_{33}}, & \dfrac{\alpha_{31}\alpha_{32}}{1-\alpha_{33}}, & -\alpha_{31}, \\
\dfrac{-\alpha_{31}\alpha_{32}}{1-\alpha_{33}}, & \dfrac{1-\alpha_{32}^2-\alpha_{33}}{1-\alpha_{33}}, & \alpha_{32}, \\
\alpha_{31}, & \alpha_{32}, & \alpha_{33}.
\end{array}$$

Substituting these values in P_{12} of (4) and integrating we find

$$y_1 = x \sin\omega, \quad y_2 = x\cos\omega, \quad y_3 = C.$$

where $\omega = \int P_{12}dt$. Substituting the values of y_1, y_2, y_3 in (6) and solving for the x's we obtain the general integral

$$(11) \qquad \begin{cases} x_1 = \dfrac{\alpha_{33}+\alpha_{31}^2-1}{1-\alpha_{33}}\, x\sin\omega - \dfrac{\alpha_{31}\alpha_{32}}{1-\alpha_{33}}\, x\cos\omega + C\alpha_{31}, \\ x_2 = \dfrac{\alpha_{31}\alpha_{32}}{1-\alpha_{33}}\, x\sin\omega + \dfrac{1-\alpha_{32}^2-\alpha_{33}}{1-\alpha_{33}}\, x\cos\omega + C\alpha_{32}, \\ x_3 = -\alpha_{31}x\sin\omega + \alpha_{32}x\cos\omega + C\alpha_{32}. \end{cases}$$

Suppose now that two integral curves are known. We shall show that the system may be reduced to the form

$$(12) \qquad \frac{dy_1}{dt} = 0 \quad \frac{dy_2}{dt} = 0, \quad \frac{dy^3}{dt} = 0.$$

Let the integral curves be C_1 and C_2. We start from an initial position, placing the extremity of the y_3-axis at a point P on C_1; on C_2 we take a point Q on the y_2-axis at a quadrant's distance from P. As t varies, P moves along C_1 while Q moves along C_2, these curves being integral curves. The axis y_1 will describe an integral curve C_3 which is uniquely determined by the motion, whenever C_1 and C_2 are known. The transformed system must now admit $y_1 = y_2 \equiv 0$ and $y_1 = y_3 \equiv 0$ as particular solutions, that is, it must reduce to the form (12), *q. e. d.*

It follows from this theorem that *the system* (3) *may be integrated without quadratures, whenever two integral curves are known.*

Analytically we proceed as follows:

Let the integral curves be

$$x_1 = \rho_1 x_3, \quad x_2 = \rho_2 x_3, \quad x_1 = \sigma_1 x_3, \quad x_2 = \sigma_2 x_3. \tag{13}$$

The transformation (6) must now satisfy the relations

$$\left\{\begin{aligned} &\rho_1\alpha_{11} + \rho_2\alpha_{12} + \alpha_{13} \equiv 0, \\ &\rho_1\alpha_{21} + \rho_2\alpha_{22} + \alpha_{23} \equiv 0, \\ &\sigma_1\alpha_{11} + \sigma_2\alpha_{12} + \alpha_{\;3} \equiv 0, \\ &\sigma_1\alpha_{31} + \sigma_2\alpha_{32} + \alpha_{33} \equiv 0, \end{aligned}\right. \tag{14}$$

Since $y_1 = y_2 \equiv 0$ whenever $x_1 = \rho_1 x_3$ and $x_2 = \rho_2 x_3$ and also $y_1 = y_3 \equiv 0$ for $x_1 = \sigma_1 x_3$ and $x_2 = \sigma_2 x_3$. These four conditions reduce to three; in fact, the distance between P and Q must equal $\sqrt{2}$ for all values of t, the initial position having been chosen as explained above. An easy calculation will show that we must have

$$\rho_1\sigma_1 + \rho_2\sigma_2 + 1 \equiv 0. \tag{15}$$

The conditions (8) may be written

$$\rho_1 = \frac{\alpha_{31}}{\alpha_{33}}, \quad \rho_2 = \frac{\alpha_{32}}{\alpha_{33}}; \quad \sigma_1 = \frac{\alpha_{21}}{\alpha_{23}}, \quad \sigma_2 = \frac{\alpha_{22}}{\alpha_{23}},$$

which show that $\alpha_{31}, \alpha_{32}, \alpha_{33}$; $\alpha_{21}, \alpha_{22}, \alpha_{23}$ are two sets of particular solutions. The remaining three cosines are now uniquely determined, so that the table of cosines will be

$$\frac{\sigma_2 - \rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\,\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, \quad \frac{\rho_1 - \sigma_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\,\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, \quad \frac{\rho_2\sigma_1 - \rho_1\sigma_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\,\sqrt{\sigma_1^2 + \sigma_2^2 + 1}},$$

$$\frac{\sigma_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}, \quad \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 + 1}},$$

$$\frac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \frac{\rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}, \quad \frac{1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}},$$

(That these quantities are direction cosines may be verified by aid of (15)).

Transforming now to new axes y_1, y_2, y_3 and reducing we find that P_{13}, P_{23} and P_{12} all vanish so that the system reduces to the form (12). The general integral is now

$$\begin{aligned} x_1 &= \frac{\sigma_2 - \rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}\, x_1 + \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}\, x_2 + \frac{\rho_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}\, x_3 , \\ x_2 &= \frac{\rho_1 - \sigma_1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}\, x_1 + \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}\, x_2 + \frac{\rho_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}\, x_3 , \\ x_3 &= \frac{\rho_2\sigma_1 - \rho_1\sigma_2}{\sqrt{\rho_1^2 + \rho_2^2 + 1}\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}\, x_1 + \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 + 1}}\, x_2 + \frac{1}{\sqrt{\rho_1^2 + \rho_2^2 + 1}}\, x_3 . \end{aligned}$$

II.

We shall now take up the study of the system

$$(1) \qquad \left\{ \begin{aligned} \frac{dx_1}{dt} &= p_{12}x_2 - p_{13}x_3 + p_{14}x_4 , \\ \frac{dx_2}{dt} &= -p_{12}x_1 + p_{23}x_3 + p_{24}x_4 , \\ \frac{dx_3}{dt} &= p_{13}x_1 - p_{23}x_2 + p_{34}x_4 , \\ \frac{dx_4}{dt} &= -p_{14}x_1 - p_{24}x_2 - p_{34}x_3 , \end{aligned} \right.$$

which has been obtained by Professor Craig.[*] Since the hypersphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \text{const.}$ remains invariant during the motion we may, as before, limit ourselves to the unit sphere. The problem of integration is then: given the six components of rotation p_{ik} around the six planes of the hexahedron of reference, to determine the motion completely. Let a curve C_1 on the hypersphere be known, and let the initial position of the axis at time t_0 be such that the x_4-axis pierces the sphere at a point P on this curve. Transforming to a new system of variable axes y_1, y_2, y_3, y_4 in such a way as to make P describe the curve C_1, as t varies, the motion of the other three axes will take place in a three-dimensional space obtained by cutting the hypersphere by a space $y_4 = 0$ passing through the origin; (this space will necessarily vary in its position with t. It follows then that after the transformation $y_1 = y_2 = y_3 \equiv 0$ must be a particular solution of the transformed system. But since a Euclidean transformation does not alter the form of the system (1) we must have

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[*] "On the displacement depending on one, two and three variables in space of 4 dimensions," Am. Jour. of Math., vol. 20.

$$(2)\qquad \begin{cases} \dfrac{dy_1}{dt} = P_{12}y_2 - P_{13}y_3\,, \\ \dfrac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3\,, \\ \dfrac{dy_3}{dt} = P_{13}y_1 - P_{23}y_2\,, \\ \dfrac{dy_4}{dt} = 0\,. \end{cases}$$

Hence the

THEOREM. *If an integral curve of the system* (1) *is known the integration of the system may be reduced to that of a similar system in three variables.*

If an integral curve C'_2 of (2) is known, it is evident that a corresponding integral curve of (1) is known and may be obtained by transforming C'_2 by the inverse of the transformation which changes the system (1) into (2). Let this transformed curve be C_2 and let us suppose it different from C_1; then two integral curves of (1) are known. But we have proved that if an integral curve of a system in three variables is known, it may be integrated by one quadrature; it follows therefore that *the original system can be integrated by one quadrature, whenever two different integral curves are known.*

Since the knowledge of two integral curves amounts to having given two sets of particular solutions, the last theorem may be stated thus:

If two sets of particular solutions of the system (1) *are known, it may be integrated by one quadrature only.*

Since the integration of (2) by Darboux's method* may be reduced to that of an ordinary Riccati equation, the above theorem may also be stated thus:
If one set of particular solutions is known the integration of (1) *may be reduced to that of a Riccati equation.*

The analytical treatment of the integration problem is similar to that employed in the case of ordinary space.

Let the integral curves be

$$(3)\qquad x_1 = \lambda_1(t)\,x_4\,,\quad x_2 = \lambda_2(t)\,x_4\,,\quad x_3 = \lambda_3(t)\,x_4\,,$$

where the λ's are known functions of t. We introduce new axes y_1, y_2, y_3, y_4 by means of the Euclidean transformation

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* Darboux, Theorie des Surfaces, vol. I, p. 22.

$$(4)\qquad \begin{cases} y_1 = \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4\,, \\ y_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \alpha_{24}x_4\,, \\ y_3 = \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 + \alpha_{34}x_4\,, \\ y_4 = \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4\,, \end{cases}$$

where the α's are the 16 direction-cosines of the new axis. Now in order that the point P shall during the motion describe the integral curve (3), we must have $y_1 = y_2 = y_3 = 0$ whenever $x_1 = \lambda_1 x_4$, $x_2 = \lambda_2 x_4$, $x_3 = \lambda_3 x_4$, or

$$(5)\qquad \begin{cases} \alpha_{11}\lambda_1 + \alpha_{12}\lambda_2 + \alpha_{13}\lambda_3 + \alpha_{14} \equiv 0\,, \\ \alpha_{21}\lambda_1 + \alpha_{22}\lambda_2 + \alpha_{23}\lambda_3 + \alpha_{24} \equiv 0\,, \\ \alpha_{31}\lambda_1 + \alpha_{32}\lambda_2 + \alpha_{33}\lambda_3 + \alpha_{34} \equiv 0\,, \end{cases}$$

which may be replaced by the following simple ones:

$$(5^1)\qquad \lambda_1 = \frac{\alpha_{41}}{\alpha_{44}}\,,\quad \lambda_2 = \frac{\alpha_{42}}{\alpha_{44}}\,,\quad \lambda_3 = \frac{\alpha_{43}}{\alpha_{44}}\,,$$

that is to say, $x_1 = \alpha_{41}$, $x_2 = \alpha_{42}$, $x_3 = \alpha_{43}$, $x_4 = \alpha_{44}$ is a set of particular solutions of the system (1). Introducing the new coordinates into the system taking into account that the α's are a system of 16 direction-cosines of which the four α_{41}, α_{42}, α_{43}, α_{44} are particular solutions, we obtain after easy reductions

$$(6)\qquad \begin{cases} \dfrac{dy_1}{dt} = P_{12}y_2 - P_{13}y_3\,, \\ \dfrac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3\,, \\ \dfrac{dy_3}{dt} = P_{13}y_1 - P_{23}y_2\,, \\ \dfrac{dy_4}{dt} = 0\,. \end{cases}$$

where P_{12}, P_{13}, P_{23} have the following values:

$$(7)\qquad \begin{cases} P_{12} = (\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12})\,p_{12} + (\alpha_{21}\alpha_{13} - \alpha_{11}\alpha_{23})\,p_{13} + (\alpha_{11}\alpha_{24} - \alpha_{21}\alpha_{14})\,p_{14} \\ \qquad + (\alpha_{12}\alpha_{23} - \alpha_{22}\alpha_{13})\,p_{23} + (\alpha_{12}\alpha_{24} - \alpha_{22}\alpha_{14})\,p_{24} + (\alpha_{24}\alpha_{14} - \alpha_{23}\alpha_{14})\,p_{34} \\ \qquad + \alpha'_{11}\alpha_{21} + \alpha'_{12}\alpha_{22} + \alpha'_{13}\alpha_{23} + \alpha'_{14}\alpha_{24}\,, \\ P_{13} = (\alpha_{12}\alpha_{31} - \alpha_{11}\alpha_{32})\,p_{12} + (\alpha_{11}\alpha_{33} - \alpha_{31}\alpha_{13})\,p_{13} + (\alpha_{31}\alpha_{14} - \alpha_{11}\alpha_{34})\,p_{14} \\ \qquad + (\alpha_{32}\alpha_{13} - \alpha_{12}\alpha_{33})\,p_{23} + (\alpha_{14}\alpha_{32} - \alpha_{12}\alpha_{34})\,p_{24} + (\alpha_{33}\alpha_{14} - \alpha_{13}\alpha_{34})\,p_{34} \\ \qquad + \alpha'_{31}\alpha_{11} + \alpha'_{33}\alpha_{13} + \alpha'_{34}\alpha_{14} + \alpha'_{32}\alpha_{12}\,, \\ P_{23} = (\alpha_{21}\alpha_{32} - \alpha_{31}\alpha_{22})\,p_{12} + (\alpha_{31}\alpha_{23} - \alpha_{33}\alpha_{21})\,p_{13} + (\alpha_{34}\alpha_{21} - \alpha_{31}\alpha_{24})\,p_{14} \\ \qquad + (\alpha_{33}\alpha_{22} - \alpha_{32}\alpha_{23})\,p_{23} + (\alpha_{34}\alpha_{22} - \alpha_{33}\alpha_{24})\,p_{24} + (\alpha_{34}\alpha_{23} - \alpha_{33}\alpha_{24})\,p_{34} \\ \qquad + \alpha_{31}\alpha'_{21} + \alpha_{32}\alpha'_{22} + \alpha_{33}\alpha'_{23} + \alpha_{34}\alpha'_{24}\,. \end{cases}$$

That the coefficients P_{14}, P_{24}, P_{34} vanish identically follows from the fact that α_{41}, α_{42}, α_{43}, α_{44} is a set of particular solutions, so that we have

$$\begin{aligned} \alpha'_{41} &= p_{12}\alpha_{42} - p_{13}\alpha_{43} + p_{14}\alpha_{44}, \\ \alpha'_{42} &= -p_{12}\alpha_{41} + p_{23}\alpha_{43} + p_{24}\alpha_{44}, \\ \alpha'_{43} &= p_{13}\alpha_{41} - p_{23}\alpha_{42} + p_{34}\alpha_{44}; \\ \alpha'_{44} &= -p_{14}\alpha_{41} - p_{24}\alpha_{42} - p_{34}\alpha_{43}; \end{aligned}$$

Substituting these values of α'_{41}, α'_{42}, α'_{43}, α'_{44} in P_{14}, P_{24} and P_{34} they vanish identically.

Since the α's may be expressed as functions of 6 parameters and since by (5) or (5^1) we have determined 4 of the 16 cosines as functions of λ_1, λ_2 and λ_3, there remain 3 arbitrary parameters which may be chosen entirely at pleasure. In the case of a system in three variables we showed how to actually calculate the simplest system of cosines in terms of the known functions ρ_1 and ρ_2. The corresponding problem for a system in four variables offers no difficulties. If we express the 16 cosines α_{ik} in terms of 6 parameters s_1, s_2, s_3, s_4, s_5, s_6 and then put $s_4 = s_5 = s_6 = 0$ we obtain the following system of cosines:*

$$(8) \quad \left\{ \begin{array}{cccc} \dfrac{1 - s_1^2 - s_2^2 - s_3^2}{\Delta}, & \dfrac{2s_1}{\Delta}, & \dfrac{-2s_2}{\Delta}, & \dfrac{2s_3}{\Delta}, \\ \dfrac{-2s_1}{\Delta}, & \dfrac{1 + s_2^2 + s_3^2 - s_1^2}{\Delta}, & \dfrac{2s_1s_2}{\Delta}, & \dfrac{-2s_1s_3}{\Delta}, \\ \dfrac{2s_2}{\Delta}, & \dfrac{2s_1s_2}{\Delta}, & \dfrac{1 + s_1^2 + s_3^2 - s_2^2}{\Delta}, & \dfrac{2s_2s_3}{\Delta}, \\ \dfrac{-2s_3}{\Delta}, & \dfrac{-2s_1s_3}{\Delta}, & \dfrac{2s_2s_3}{\Delta}, & \dfrac{1 + s_1^2 + s_2^2 - s_3^2}{\Delta}, \end{array} \right.$$

where $\Delta = 1 + s_1^2 + s_2^2 + s_3^2$. Now since α_{41}, α_{42}, α_{43}, α_{44} is a set of particular solutions we have

$$\frac{2s_3}{\Delta} = \alpha_{41}, \quad \frac{-2s_1s_3}{\Delta} = \alpha_{42}, \quad \frac{2s_2s_3}{\Delta} = \alpha_{43}, \quad \frac{1 + s_1^2 + s_2^2 - s^2}{\Delta} = \alpha_{44};$$

Solving these equations for s_1, s_2 and s_3 we find

$$s_1 = \frac{\alpha_{42}}{\alpha_{41}}, \quad s_2 = \frac{-\alpha_{43}}{\alpha_{41}}, \quad s_3 = \frac{\alpha_{44} - 1}{\alpha_{41}};$$

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* See Pascal, Die Determinanten, pp. 160–163.

the table (8) may now be written

$$\begin{array}{cccc}
\dfrac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}, & \dfrac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}, & \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}, & -\alpha_{41}, \\
\dfrac{-\alpha_{42}\alpha_{41}}{1-\alpha_{44}}, & \dfrac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}, & \dfrac{-\alpha_{42}\alpha_{43}}{1-\alpha_{44}}, & \alpha_{42}, \\
\dfrac{-\alpha_{43}\alpha_{41}}{1-\alpha_{44}}, & \dfrac{-\alpha_{42}\alpha_{43}}{1-\alpha_{44}}, & \dfrac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}, & \alpha_{43}, \\
\alpha_{41}, & \alpha_{42}, & \alpha_{43}, & \alpha_{44},
\end{array}$$

and we thus arrive at the following orthogonal transformation:

$$(9)\quad \begin{cases}
y_1 = \dfrac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}x_1 + \dfrac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}x_2 + \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}x_3 - \alpha_{41}x_4, \\
y_2 = \dfrac{-\alpha_{41}\alpha_{42}}{1-\alpha_{44}}x_1 + \dfrac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}x_2 - \dfrac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}x_3 + \alpha_{42}x_4, \\
y_3 = \dfrac{-\alpha_{43}\alpha_{41}}{1-\alpha_{44}}x_1 - \dfrac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}x_2 + \dfrac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}x_3 + \alpha_{43}x_4, \\
y_4 = \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4,
\end{cases}$$

by means of which the system (1) *is transformed into the system* (6).

Now suppose we know a set of particular solutions of (6), or, to use geometrical language, an integral curve on the sphere $y_1^2+y_2^2+y_3^2=1$, $y_4=0$. Since the first three equations (6) are independent of the last, we consider these equations as defining a rotation of the sphere $y_1^2+y_2^2+y_3^2=\text{const.}$ around a fixed point 0. This case has been treated in the first part of this paper, where we found that such a system is integrable by one quadrature, a set of particular solutions being known.

Now I say that *if a set of particular solutions of* (6) *is known, a set of particular solutions of* (1) *is known and conversely.* The proof is immediate, for, let $\beta_{31}, \beta_{32}, \beta_{33}$ 0, be such a set; substituting these for y_1, y_2, y_3, y_4 respectively in (9) and solving for the x's we obtain

$$(10)\quad \begin{cases}
x_1^0 = \dfrac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}\beta_{31} - \dfrac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}}\beta_{32} - \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\beta_{33}, \\
x_2^0 = \dfrac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}\beta_{31} + \dfrac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{32} - \dfrac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}\beta_{33}, \\
x_3^0 = \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\beta_{31} - \dfrac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\beta_{32} + \dfrac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{33}, \\
x_4^0 = -\alpha_{41}\beta_{31} + \alpha_{42}\beta_{32} + \alpha_{43}\beta_{33},
\end{cases}$$

which are easily seen to satisfy the relation $x_1^{0^2} + x_2^{0^2} + x_3^{0^2} + x_4^{0^2} = 1$. Morever, since $y_4 = 0$ is a particular solution of the last of equations (6) we have from the last of equations (9)

$$y_4 = 0 = \alpha_{41}x_1^0 + \alpha_{42}x_2^0 + \alpha_{43}x_3^0 + \alpha_{44}x_4^0,$$

which means that $x_1^0, x_2^0, x_3^0, x_4^0$ are four direction-cosines on the hypersphere.

Conversely, suppose given two sets of particular solutions of (1), viz.:

$$\alpha_{41}\alpha_{42}\alpha_{43}\alpha_{44}\,;\; x_1^0,\, x_2^0,\, x_3^0,\, x_4^0,$$

and if we transform (1) by means of (9) into the system (6), this system will have for particular solutions $\beta_{31}, \beta_{32}, \beta_{33}, 0$ obtained by substituting $x_1^0, x_2^0, x_3^0, x_4^0$ for the x's in (9), that is, we will have

$$y_1 = \beta_{31},\; y_2 = \beta_{32},\; y_3 = \beta_{33},\; y_4 = 0,\; q.e.d.$$

We are now ready to complete the problem. Transforming (6) to a system of new axes z_1, z_2, z_3 by means of the transformation

$$(11)\qquad \begin{cases} z_1 = \dfrac{\beta_{33} + \beta_{31}^2 - 1}{1 - \beta_{33}}\, y_1 + \dfrac{\beta_{31}\beta_{32}}{1 - \beta_{33}}\, y_2 - \beta_{31}y_3, \\[2mm] z_2 = \dfrac{-\beta_{31}\beta_{32}}{1 - \beta_{33}}\, y_1 + \dfrac{1 - \beta_{33} - \beta_{32}^2}{1 - \beta_{33}}\, y_2 + \beta_{32}y_3, \\[2mm] z_3 = \beta_{31}y_1 + \beta_{32}y_2 + \beta_{33}y_3, \\ z_4 = y_4, \end{cases}$$

we obtain a new system of the form

$$(12)\qquad \frac{dz_1}{dt} = \bar{P}_{12}z_2, \quad \frac{dz_2}{dt} = -\bar{P}_{12}z_1, \quad \frac{dz_3}{dt} = 0, \quad \frac{dz_4}{dt} = 0,$$

which may be integrated by one quadrature. The general integral of (1) may now be easily obtained. The general integral of (6) is, (compare (11) part I)

$$\begin{aligned} y_1 &= \frac{\beta_{33} + \beta_{31}^2 - 1}{1 - \beta_{33}}\, \kappa \sin\bar{\omega} - \frac{\beta_{31}\beta_{32}}{1 - \beta_{33}}\, \kappa \cos\bar{\omega} + C\beta_{31}, \\ y_2 &= \frac{\beta_{31}\beta_{32}}{1 - \beta_{33}}\, \kappa \sin\bar{\omega} + \frac{1 - \beta_{33} - \beta_{32}^2}{1 - \beta_{33}}\, \kappa \cos\bar{\omega} + C\beta_{32}, \\ y_3 &= -\beta_{31}\, \kappa \sin\bar{\omega} + \beta_{32}\, \kappa \cos\bar{\omega} + C\beta_{33}, \\ y_4 &= \text{const.} \end{aligned}$$

where $\bar{\omega} = \int \bar{P}_{12}dt$. Substituting these values in (9) and solving for the x's we obtain the general integral

$$
(13)\quad \left\{
\begin{aligned}
x_1 = {} & \left\{\frac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}\cdot\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}-\frac{\alpha_{41}\alpha_{42}\beta_{31}\beta_{32}}{(1-\alpha_{44})(1-\beta_{33})}\right.\\
& \qquad\qquad\qquad\qquad\left.+\frac{\alpha_{43}\alpha_{41}\beta_{31}}{1-\alpha_{44}}\right\}\varkappa\sin\bar{\omega}\\
& -\left\{\frac{\alpha_{41}^2+\alpha_{44}-1}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}+\frac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}}\cdot\frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}}\right.\\
& \qquad\qquad\qquad\qquad\left.-\frac{\alpha_{43}\alpha_{41}\beta_{32}}{1-\alpha_{44}}\right\}\varkappa\cos\bar{\omega}\\
& +\ cx_1^0+\text{const. }\alpha_{41}.\\
x_2 = {} & \left\{\frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta}+\frac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\right.\\
& \qquad\qquad\qquad\qquad\left.+\frac{\alpha_{42}\alpha_{43}\beta_{31}}{1-\alpha_{44}}\right\}\varkappa\sin\bar{\omega}\\
& -\left\{\frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}-\frac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}\cdot\frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}}\right.\\
& \qquad\qquad\qquad\qquad\left.+\frac{\alpha_{42}\alpha_{43}\beta_{32}}{1-\alpha_{44}}\right\}\varkappa\cos\bar{\omega}\\
& +\ cx_2^0+\text{const. }\alpha_{42}.\\
x_3 = {} & \left\{\frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}-\frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\right.\\
& \qquad\qquad\qquad\qquad\left.-\frac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{31}\right\}\varkappa\sin\bar{\omega}\\
& -\left\{\frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\cdot\frac{\beta_{31}\beta_{33}}{1-\beta_{33}}+\frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}\cdot\frac{1-\beta_{33}\beta_{32}^2}{1-\beta_{33}}\right.\\
& \qquad\qquad\qquad\qquad\left.-\frac{1-\alpha_{43}^2-\alpha_{44}}{1-\alpha_{44}}\beta_{32}\right\}\varkappa\cos\bar{\omega}\\
& +\ cx_3^0+\text{const. }\alpha_{43}.\\
x_4 = {} & -\left\{\alpha_{41}\cdot\frac{\beta_{33}+\beta_{32}^2-1}{1-\beta_{33}}-\frac{\alpha_{42}\beta_{31}\beta_{32}}{1-\beta_{33}}+\alpha_{43}\beta_{31}\right\}\varkappa\sin\bar{\omega}\\
& +\left\{\frac{\alpha_{41}\beta_{31}\beta_{32}}{1-\beta_{33}}-\frac{\alpha_{42}\beta_{31}\beta_{32}}{1-\beta_{33}}+\alpha_{43}\beta_{32}\right\}\varkappa\cos\bar{\omega}\\
& +\ cx_4^0+\text{const. }\alpha_{44}.
\end{aligned}
\right.
$$

In these formulæ the values of β_{31}, β_{32}, β_{33} are to be replaced by their values in terms of x_1^0, x_2^0, x_3^0, x_4^0 obtained from equations (10). The determination of the angle ω is the only quadrature involved, otherwise the work has been purely algebraic.*

---

*It should be noticed that the system of integrals (13) contain 4 constants of integration: κ, C, const. and a fourth is obtained in the quadrature $\int P_{12}dt$.

It is now comparatively easy to obtain an orthogonal transformation which shall transform (1) directly into (12). In fact, combining (9) and (11) we obtain after somewhat lengthy reductions:

$$
(14)\left\{\begin{aligned}
z_1 = {} & \left\{\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}\cdot\frac{\alpha_{44}+\alpha_{41}^2-1}{1-\alpha_{44}}-\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\cdot\frac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}}+\frac{\beta_{31}\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\right\}x_1 \\
& +\left\{\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}\cdot\frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}+\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\cdot\frac{1-\alpha_{44}^2-\alpha_{44}}{1-\alpha_{44}}+\frac{\beta_{31}\alpha_{42}\alpha_{43}}{1-\alpha_{44}}\right\}x_2 \\
& +\left\{\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}\cdot\frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}-\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\cdot\frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}-\frac{\beta_{31}(1-\alpha_{43}^2-\alpha_{44})}{1-\alpha_{44}}\right\}x_3 \\
& -\left\{\frac{\beta_{33}+\beta_{31}^2-1}{1-\beta_{33}}\cdot\alpha_{41}-\frac{\beta_{31}\beta_{33}}{1-\beta_{33}}\cdot\alpha_{42}+\beta_{31}\alpha_{43}\right\}x_4. \\
z_2 = {} & -\left\{\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\cdot\frac{\alpha_{44}^2+\alpha_{41}^2-1}{1-\alpha_{44}}+\frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}}\cdot\frac{\alpha_{41}\alpha_{42}}{1-\alpha_{44}}+\frac{\beta_{32}\alpha_{43}\alpha_{41}}{1-\alpha_{44}}\right\}x_1 \\
& -\left\{\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\cdot\frac{\alpha_{42}\alpha_{41}}{1-\alpha_{44}}-\frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}}\cdot\frac{1-\alpha_{42}^2-\alpha_{44}}{1-\alpha_{44}}+\frac{\beta_{32}\alpha_{43}\alpha_{42}}{1-\alpha_{44}}\right\}x_2 \\
& -\left\{\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\cdot\frac{\alpha_{43}\alpha_{41}}{1-\alpha_{44}}+\frac{1-\beta_{33}-\beta_{32}^2}{1-\beta_{33}}\cdot\frac{\alpha_{42}\alpha_{43}}{1-\alpha_{44}}-\frac{\beta_{32}(1-\alpha_{43}^2-\alpha_{44})}{1-\alpha_{44}}\right\}x_3 \\
& +\left\{\frac{\beta_{31}\beta_{32}}{1-\beta_{33}}\cdot\alpha_{41}+\frac{1-\beta_{33}-\beta_{32}^3}{1-\beta_{33}}\cdot\alpha_{42}+\beta_{32}\alpha_{43}\right\}x_4. \\
z_3 = {} & x_1^0x_1+x_2^0x_2+x_3^0x_3+x_4^0x_4, \\
z_4 = {} & \alpha_{41}x_1+\alpha_{42}x_2+\alpha_{43}x_3+\alpha_{44}x_4.
\end{aligned}\right.
$$

In the first two equations $\beta_{31}, \beta_{32}, \beta_{33}$ have the same values as in (13). This transformation exhibits a system of direction-cosines depending on the 8 parameters $\alpha_{41}, \alpha_{42}, \alpha_{43}, \alpha_{44}, x_1^0, x_2^0, x_3^0, x_4^0$ between which 3 relations exist. We have thus incidentally solved the problem: *of a set of* 16 *direction-cosines* 8 *are given; it is required to calculate the remaining* 8 *in terms of the given parameters. The solution of this problem is equivalent to reducing the integration of* (1) *to a single quadrature, when two sets of particular solutions are known.* It should be noticed that this problem can be solved in an infinite number of ways, the simplest set of cosines being obtained by putting $s_4 = s_5 = s_6 = 0$. *There exist then* ∞^8 *Euclidean transformations which will effect the reduction of the system* (1) *to the form*

$$
\frac{dz_1}{dt} = P_{12}z_2, \quad \frac{dz_2}{dt} = -P_{12}z_1, \quad \frac{dz_3}{dt} = 0, \quad \frac{dz_3}{dt} = 0.
$$

The case where three sets of particular solutions are known (or three integral curves) requires no special consideration; a Euclidean transformation may then be found which will transform (1) into the system

$$\frac{dy_1}{dt}=0\,,\quad \frac{dy_2}{dt}=0\,,\quad \frac{dy_3}{dt}=0\,,\quad \frac{dy_4}{dt}=0\,.$$

which requires no quadrature. If $\alpha_{41}, \alpha_{42}, \alpha_{43}, \alpha_{44}; \alpha_{31}, \alpha_{32}, \alpha_{33}, \alpha_{34}; \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}$ be the given sets, the transformation may be written down at once as follows:

$$\begin{aligned} y_1 &= D_{11}x_1 + D_{12}x_2 + D_{13}x_3 + D_{14}x_4\,, \\ y_2 &= \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \alpha_{24}x_4\,; \\ y_3 &= \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3 + \alpha_{34}x_4\,, \\ y_4 &= \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4\,, \end{aligned}$$

where $D_{11}, D_{12}, D_{13}, D_{14}$ are the minors of $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}$ in the determinant of the 16 cosines $|\, \alpha_{11}, \alpha_{22}, \alpha_{33}, \alpha_{44} \,|$.

III.

There remains yet the problem of generalizing the preceding theory of integration. Mr. Hatzidakis has obtained the general system defining the rotation of a rigid system in n-dimensional space around a fixed point.* This system may with slight modifications affecting only the signs of the coefficients p_{ik} be be written †

$$(1)\qquad \left\{ \begin{aligned} \frac{dx_1}{dt} &= \quad p_{12}x_2 + p_{13}x_3 + \;\ldots\ldots\ldots\ldots\; + p_{12}x_n\,, \\ \frac{dx_2}{dt} &= -p_{12}x_1 + p_{23}x_3 + \;\ldots\ldots\ldots\ldots\; + p_{2n}x_n\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{dx_n}{dt} &= -p_{1n}x_1 - p_{2n}x_2 - \;\ldots\ldots\ldots\ldots\; - p_{n-1n}x_{n-1}. \end{aligned} \right.$$

It will be easy to prove as in the case of four dimensions that *the integration of this system may be reduced to that of a similar system, but of one dimension less, whenever a set of particular solutions is known.* The given integral curve being C_1 we transform to a new set of axes $y_1, y_2, \ldots . y_n$ in such a way that the y_n-axis pierces the hypersphere $\Sigma x_n^2 = 1$ at a point P on the given integral curve. In the transformed system $y_1 = y_2 = y_3 = \dots\dots y_{n-1} = 0$ must be a particular solution, so that the last equation reduces to the form $\frac{dy_n}{dt} = 0$, and the first $n - 1$ equations will define a rotation of the sphere $y_1^2 + y_2^2 + \dots\dots y_{n-1}^2 = C$, $y_n = 0$ about the fixed point 0. In general, if m integral curves are known, we may transform the system by means of a Euclidean transformation in such a way that y_n describes C_1, y_{n-1} describes $C_2 \dots\dots$ and y_{n-m} describes $C_m \dots\dots$ The extremities of the remaining axis will move along paths on the $n - m$ dimensional sphere $y_1^2 + y_2^2 + \dots\dots y_{n-m}^2 = 1$, $y_{n-m+1} = y_{n-m+2} = \dots\dots y_n = 0$, and the system (1) will be reduced to the form

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* Am. Jour. of Math., vol. 23.

† See my article "On Nullsystems in Space of 5 Dimensions and their Relation to Ordinary Space," Am. Jour. of Math., vol. 25, p. 105. The p's in this article are constants, while here they are given functions of t; moreover, n is not restricted to an odd integer; n may be even or odd.

$$(2)\quad \left\{\begin{array}{rcl}
\frac{dy_1}{dt} & = & P_{12}y_2 + P_{13}y_3 + \dots\dots\dots + P_{1n-m}y_{n-m}, \\
\frac{dy_2}{dt} & = & -P_{12}y_1 + P_{23}y_3 + \dots\dots\dots + P_{2n-m}y_{n-m}, \\
\dots\dots & \dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots & \dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{dy_{n-m}}{dt} & = & -P_{1n-m}y_1 - P_{2n-m}y_2 - \dots\dots\dots - P_{n-m-1,n-m}y_{n-m-1}, \\
\frac{dy_{n-m+1}}{dt} & = & 0, \\
\dots\dots & \dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots & \dots & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{dy_n}{dt} & = & 0.
\end{array}\right.$$

In particular, *if $n - 2$ integral curves are known the system may be reduced to the form*

$$\frac{dy_1}{dt} = P_{12}y_2, \quad \frac{dy_2}{dt} = -P_{12}y_1, \quad \frac{dy_3}{dt} = 0, \quad \dots\dots\dots\dots, \quad \frac{dy_n}{dt} = 0,$$

and may therefore be integrated by one quadrature.

We shall show how this reduction may be effected by means of a series of Euclidean transformations. The problem consists in determining a set of n^2 cosines in terms of a given set of particular solutions, n being any given integer > 4.

Let the integral curve be

$$x_1 = \rho_1 x_n, \quad x_2 = \rho_2 x_n, \quad \dots\dots\dots, \quad x_{n-1} = \rho_{n-1} x_n.$$

Since on the unit hypersphere $x_1^2 + x_2^2 + \dots\dots x_n^2 = 1$, this is equivalent to the knowledge of a set of particular solutions

$$\frac{\rho_1}{\sqrt{1 + \Sigma\rho_{n-1}^2}},\quad \frac{\rho_2}{\sqrt{1 + \Sigma\rho_{n-1}^2}},\quad \dots\dots\dots\dots \frac{\rho_{n-1}}{\sqrt{1 + \Sigma\rho_{n-1}^2}},\quad \frac{1}{\sqrt{1 + \Sigma\rho_{n-1}^2}},$$

We employ the orthogonal transformation

$$(3)\qquad \begin{cases} y_1 = \alpha_{11}x_1 + \alpha_{12}x_2 + \dots\dots\dots + \alpha_{1n}x_n, \\ y_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \dots\dots\dots + \alpha_{2n}x_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n = \alpha_{n1}x_1 + \alpha_{n2}x_2 + \dots\dots\dots + \alpha_{nn}\alpha_n. \end{cases}$$

In the same way as before in the case of $n = 4$, $y_1 = y_2 = \dots\dots = y_{n-1}$ must vanish, whenever $x_1 = \rho_1 x_n$, $x_2 = \rho_2 x_n$, $\dots\dots x_{n-1} = \rho_{n-1}x_n$, and hence the following conditions must be satisfied:

$$\begin{aligned} &\alpha_{11}\rho_1 + \alpha_{12}\rho_2 + \dots\dots\dots + \alpha_{1n-1}\rho_{n-1} + \alpha_{1n} \equiv 0, \\ &\alpha_{21}\rho_1 + \alpha_{22}\rho_2 + \dots\dots\dots + \alpha_{2n-1}\rho_{n-1} + \alpha_{2n} \equiv 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\alpha_{n-11}\rho_1 + \alpha_{n-12}\rho_2 + \dots\dots\dots + \alpha_{n-1n-1}\rho_{n-1} + \alpha_{n-1n} \equiv 0, \end{aligned}$$

which $n - 1$ conditions may be reduced to the form

$$(4)\qquad \rho_1 = \frac{\alpha_{n1}}{\alpha_{nn}},\quad \rho_2 = \frac{\alpha_{n2}}{\alpha_{nn}},\quad \dots\dots\dots \rho_{n-1} = \frac{\alpha_{nn-1}}{\alpha_{nn}},$$

that is, $\alpha_{n1}, \alpha_{n2}, \dots\dots \alpha_{nn}$ must be the given set of particular solutions. In order to obtain a transformation that will reduce the system (1), which we shall denote by Λ_n, to a system Λ_{n-1}, we must find a table of cosines α_{ik} expressed in terms of the n cosines α_{ni}, $(i = 1, 2 \dots\dots n)$. Since these n cosines are determined as functions of $\rho_1, \rho_2 \dots\dots \rho_{n-1}$, there remain $\frac{n.n-1}{2} - (n-1)$ parameters which may be disposed of at will. Let the $\frac{n.n-1}{2}$ parameters be denoted by $s_1, s_2, \dots\dots s_{\frac{n.n-1}{2}}$. We now put all except the first $n - 1$ of these equal to zero and calculate the n^2 direction-cosines in terms of these parameters by the ordinary method. (Pascal, Die Determinanten, pp. 160–161). We find after easy calculations the following table,

$$\frac{1-s_1^2-s_2^2-\dots\, s_{n-1}^2}{\Delta},\quad \frac{2s_1}{\Delta},\quad \frac{-2s_2}{\Delta},\quad \dots\dots\dots \frac{(-1)^n 2s_{n-1}}{\Delta},$$

$$\frac{-2s_1}{\Delta},\quad \frac{1+s_2^2+\dots+s_{n-1}^2-s_1^2}{\Delta},\quad \frac{2s_1s_2}{\Delta},\quad \dots\dots\dots \frac{(-1)^{n-1}2s_1s_{n-1}}{\Delta},$$

$$\frac{2s_2}{\Delta},\quad \frac{2s_1s_2}{\Delta},\quad \frac{1+s_2^2+s_3^2+s_4^2+\dots+s_{n-1}^2-s_2^2}{\Delta},\quad \dots\dots\dots \frac{(-1)^{n-2}2s_2s_{n-1}}{\Delta},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{(-1)^{n-1}2s_{n-1}}{\Delta},\quad \frac{(-1)^{n-1}2s_1s_{n-1}}{\Delta},\quad \frac{(-1)^{n-2}2s_2s_{n-1}}{\Delta},\quad \dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots \frac{1+s_1^2+s_2^2+\dots+s_{n-1}^2-s_{n-1}^2}{\Delta},$$

where $\Delta = 1 + \sum_1^{n-1} s_n^2$. We have then

$$\frac{(-1)^{n-1}2s_{n-1}}{\Delta} = \alpha_{n1},\quad \frac{(-1)^{n-1}2s_1s_{n-1}}{\Delta} = \alpha_{n2},\quad \dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots \frac{1+s_1^2+\dots+s_{n-2}^2-s_{n-1}^2}{\Delta} = \alpha_{nn},$$

solving which for $s_1, s_2, \dots\, s_{n-1}$ we obtain

$$s_1 = \frac{\alpha_{n2}}{\alpha_{n1}};\quad s_2 = \frac{-\alpha_{n3}}{\alpha_{n1}},\quad s_3 = +\frac{\alpha_{n4}}{\alpha_{n1}};\quad \dots\dots\dots\quad s_{n-1} = \frac{\alpha_{nn}-1}{\alpha_{n1}},$$

and we have the following transformation:

$$(5)\quad \begin{cases} y_1 = \dfrac{\alpha_{n1}^2+\alpha_{nn}-1}{1-\alpha_{nn}}x_1 + \dfrac{\alpha_{n2}\alpha_{n1}}{1-\alpha_{nn}}x_2 + \dots + \dfrac{\alpha_{nn-1}\alpha_{n1}}{1+\alpha_{nn}}x_{n-1} + (-1)^n\alpha_{n1}x_n, \\ y_2 = -\dfrac{\alpha_{n2}\alpha_{n1}}{1-\alpha_{nn}}x_1 + \dfrac{1-\alpha_{n2}^2-\alpha_{nn}}{1-\alpha_{nn}}x_2 + \dots - \dfrac{\alpha_{nn-1}\alpha_{n2}}{1+\alpha_{nn}}x_{n-1} + (-1)^{n-1}\alpha_{n2}x_n, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n = \alpha_{n1}x_1 + \alpha_{n2}x_2 + \dots\dots\dots\dots\, \alpha_{nn-1}x_3 + \alpha_{nn}x_4, \end{cases}$$

which transforms (1) into the form

$$(6)\qquad \left\{\begin{array}{l} \dfrac{dy_1}{dt} = \quad P_{12}y_2 + P_{13}y_3 + \cdots\cdots + P_{1n-1}y_{n-1}, \\ \dfrac{dy_2}{dt} = -P_{12}y_1 + P_{23}y_3 + \cdots\cdots + P_{2n-1}y_{n-1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{dy_{n-1}}{dt} = -P_{1n-1}y_1 - P_{2n-2}y_2 - \cdots\cdots - P_{n-2n-1}y_{n-2}, \\ \dfrac{dy_n}{dt} = \quad 0. \end{array}\right.$$

By a method analogous to the one we employed in the case of $n = 4$ we may prove that if a set of particular solutions of (6) is known, a set of solutions of (1) is also known and conversely. Treating the system (6) in the same manner as (1) we may reduce it to another system the integration of which is equivalent to that of a system of dimension $n - 2$. The Euclidean transformation transforming Λ_n into a system Λ_{n-2} is found by combining the two transformations transforming Λ_n into Λ_{n-1} and Λ_{n-1} into Λ_{n-2}. In general, calling the successive systems Λ_n, Λ_{n-1}, ... Λ_{n-m}, and the corresponding transformations $S_n, S_{n-1}, \ldots S_{n-m+1}$, we may by means of a transformation $S = S_n S_{n-1} \ldots S_{n-m+1}$ transform Λ_n into Λ_{n-m}. In particular, if $m = n - 2$ the transformation $S = S_n S_{n-1} \ldots\ldots S_3$ will transform Λ_n into the system Λ_2 or,

$$\frac{dz_1}{dt} = P_{12}z_2, \quad \frac{dz_2}{dt} = -P_{12}z_1, \quad \frac{dz_3}{dt} = 0, \quad \cdots\cdots \quad \frac{dz_n}{dt} = 0,$$

which may be integrated by one quadrature.

It appears from the preceding investigations that the problem of reduction of Λ_n to Λ_{n-m} amounts to this: *Given $m.n$ cosines as functions of t, to calculate the remaining $n^2 - nm$ in terms of the given functions.* This is a purely algebraic problem, involving only the combination of $m - 1$ known transformations S_n, S_{n-1}, ... S_{n-m+1} where S_n is of the form (5). There exist $\infty^{\frac{n-1,n-2}{2}}$ transformations which will transform Λ_n into Λ_{n-1} of which the transformation (5) is the simplest, corresponding, as it does, to the case where $\frac{n-2, n-1}{2}$ of the parameters $S_1 \ldots\ldots S_{\frac{n.n-1}{2}}$ have been put equal to zero. The transformation

$S = S_n, S_{n-1} \ldots . S_{n-m+1}$ contains $(m-1)n - \frac{m(1+m)}{2}$ determinate parameters which are given functions of t; it follows therefore that in general there exist $\frac{n \, . \, n-1}{2} - \left\{ n \, . \, m - 1 - \frac{m.(1+m)}{2} \right\} = \frac{(n-m)(n-m-1)}{2}$ arbitrary parameters, so that *there are* $\infty^{\frac{(n-m)(n-m-1)}{2}}$ *transformations which will reduce* Λ_n *to* Λ_{n-m}, *m sets of particular solutions being known.* In particular, if $m = n-1$ we find, since now $(n-m)(n-m-1) = 0$, that the transformation is uniquely determined by the $n-1$ given sets of solutions, which is indeed obvious, since $n-1$ sets of direction-cosines determine the remaining set of n cosines. In the transformation S all the $\frac{(n-m)(n-m-1)}{2}$ arbitrary parameters have been put equal to zero which insures that S is the simplest transformation possible.

If we compare the determinant of the system (14), (part II) with the determinant of the systems (9) and (11), we observe that the first is the product of the other two. In general, if we write the successive determinants of the transformations $S_n, S_{n-1} \ldots . S_{n-m+1}$ as follows:

$$D_n = \begin{vmatrix} \alpha_{11}^{(n)} \alpha_{12}^{(n)} & \ldots . & \alpha_{1n}^{(n)} \\ \alpha_{21}^{(n)} \alpha_{22}^{(n)} & \ldots . & \alpha_{2n} \\ \vdots \quad \vdots & & \vdots \\ \alpha_{n1}^{(n)} \alpha_{n2}^{(n)} & \ldots . & \alpha_{nn}^{(n)} \end{vmatrix}, \quad D_{n-1} = \begin{vmatrix} \alpha_{11}^{(n-1)} \alpha_{12}^{(n-1)} & \ldots . & \alpha_{1n-1}^{(n-1)} & 0 \\ \alpha_{21}^{(n-1)} \alpha_{22}^{(n-1)} & \ldots . & \alpha_{2n-1}^{(n-1)} & 0 \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ \alpha_{n-11}^{(n-1)} \alpha_{n-12}^{(n-1)} & \ldots . & \alpha_{n-1n-1}^{(n-1)} & 0 \\ 0 \quad 0 & \ldots . & 0 & 1 \end{vmatrix},$$

$$\ldots\ldots\ldots\ldots D_{n-m+1} = \begin{vmatrix} \alpha_{11}^{(n-m+1)} \alpha_{12}^{(n-m+1)} & \ldots . & \alpha_{1n-m+1}^{(n-m+1)} & 0 \ldots . 0 \\ \alpha_{21}^{(n-m+1)} \alpha_{22}^{(n-m+1)} & \ldots . & \alpha_{2n-m+1}^{(n-m+1)} & 0 \ldots . 0 \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_{n-m+11}^{(n-m+1)} \alpha_{n-m+12}^{(n-m+1)} & \ldots . & \alpha_{n-m+1n-m+1}^{(n-m+1)} & 0 \ldots . 0 \\ 0 \quad 0 & \ldots . & 0 & 1 \ldots . 0 \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 \quad 0 & \ldots . & 0 \ldots\ldots & \ldots\ldots 1 \end{vmatrix},$$

and from the product $D = D_n D_{n-1} \ldots . D_{n-m+1}$ we obtain the determinant of the direction-cosines in the transformation S which transforms Λ_n into Λ_{n-m}. Let now m sets of direction-cosines be known particular solutions and let these be written

$$x_{n1}^0,\quad x_{n2}^0 \cdots\cdots x_{nn}^0;\quad x_{n-11}^0,\quad x_{n-12}^0,\ \cdots\cdots,\ x_{n-1n}^0,\ \cdots\cdots\cdots, \\ \cdots\cdots x_{n-m+11}^0, x_{n-m+12}^0 \cdots\cdots x_{n-m+1n}^0.$$

In the transformation S_n which transforms Λ_n into Λ_{n-1} the last row of direction-cosines are identical with the first set of given solutions (see (5)), and all the remaining cosines in D_n are expressible in terms of these solutions. In D_{n-1} the $n-1^{st}$ row are particular solutions of the system Λ_{n-1} and the remaining cosines are expressible in terms of these; but by means of the transformation S_n^{-1} we may express $\alpha_{n-11}^{(n-1)}\ \alpha_{n-12}^{(n-1)} \cdots\cdot \alpha_{n-1n-1}^{(n-1)}, 0$ in terms of the second set of solutions $x_{n-11}^0\ x_{n-12}^0 \cdots\cdot x_{n-1n}^0$, so that all the elements of D_{n-1} will be expressed in terms of these functions. Continuing the reasoning in this way, we find that the elements of D_{n-m+1} may all be expressed in terms of the last set of given solutions, viz.: $x_{n-m+11}^0, x_{n-m+12}^0 \cdots\cdot x_{n-m+1n}^0$. It follows that the product D will be an orthogonal determinant of the form

$$D = \begin{vmatrix} A_{11} & A_{12} \cdots\cdots & A_{1n} \\ A_{21} & A_{22} \cdots\cdots & A_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ A_{n-m1} & A_{n-m2} \cdots\cdots & A_{n-mn} \\ x_{n-m+11}^0 & x_{n-m+12}^0 \cdots & x_{n-m+1n}^0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x_{n1}^0 & x_{n2}^0 \cdots\cdots & x_{nn}^0 \end{vmatrix}$$

where the A's are functions of the x's in the last m rows. If $m = n-1$ the determinant reduces to the form

$$D = \begin{vmatrix} A_{11} & A_{12} \cdots\cdot & A_{1n} \\ x_{21}^0 & x_{22}^0 \cdots\cdot & x_{2n}^0 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ x_{n1}^0 & x_{n2}^0 \cdots\cdot & x_{nn}^0 \end{vmatrix},$$

where $A_{11} \cdots\cdot A_{1n}$ are the minors of $x_{11}^0,\ x_{12}^0 \cdots\cdot x_{1n}^0$ in the determinant $|x_{11}^0, x_{22}^3, x_{33}^0 \cdots\cdot x_{nn}^0|$.

If $m = n - 2$, we have the determinant

$$D = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ x^0_{31} & x^0_{32} & \dots & x^0_{3n} \\ \vdots & & & \\ x^0_{n1} & x^0_{n2} & \dots & x^0_{nn} \end{vmatrix},$$

and the general integral of (1) may be written down at once. We have

$$\begin{aligned} x_1 &= A_{11}x \sin \bar{\omega} + A_{21}x \cos \omega + C_1 x^0_{31} + C_2 x^0_{41} + \dots\dots + C_{n-2} x^0_{n1}, \\ x_2 &= A_{12}x \sin \omega + A_{22}x \cos \omega + C_1 x^0_{32} + C_2 x^0_{42} + \dots\dots + C_{n-2} x^0_{n2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n &= A_{1n}x \sin \omega + A_{2n}x \cos \omega + C_1 x^0_{3n} + C_2 x^0_{4n} + \dots\dots + C_{n-2} x^0_{nn}, \end{aligned}$$

where $\omega = \int P_{12}dt$, and $C_1, C_2 \,..\, C_{n-2}$ are integration constants. $A_{11}, A_{12} \,..\, A_{1n}$ and $A_{21}, A_{22} \dots A_{2n}$ are all calculable in terms of the $n - 2$ given sets of particular solutions by a purely algebraic process as was explained above. For $m = n - 1$ we have the general integrals:

$$\begin{aligned} x_1 &= C_1 A_{11} + C^2 x^0_{21} + C_3 x^0_{31} + \dots\dots\ C_n x^0_{n1}, \\ x_2 &= C_1 A_{12} + C_2 x^0_{22} + C_3 x^0_{32} + \dots\dots\ C_n x^0_{n2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n &= C_1 A_{1n} + C_2 x^0_{2n} + \dots\dots\dots\ C_n x^0_{nn}, \end{aligned}$$

where $A_{11}, A_{12} \dots A_{1n}$ are the above-mentioned minors and $C_1, C_2 \dots C_n$ are the n integration constants.

The integration problem is thus reduced to its simplest form. It may be stated thus: *If m sets of particular solutions of the system Λ_n be known, the integration may be reduced to that of a system Λ_{n-m}.* If this last system be transformed by means of the transformation

$$\frac{y_1}{y_{n-m}} = \omega_1, \quad \frac{y_2}{y_{n-m}} = \omega_2, \quad \dots\dots \quad \frac{y_{n-m-1}}{y_{n-m}} = \omega_{n-m-1},$$

into a generalized Riccati system in $n - m - 1$ variables, the above statement may be put in the form: *If m sets of particular solutions of Λ_n be known, the system may be reduced to a system of Riccati equations in $n - m - 1$ variables.*

For $m = n - 2$ we find the Riccati equation

$$\frac{d\omega}{dt} = P_{12}(1 + \omega^2),$$

which is integrable by one quadrature.

It should be noticed that the Riccati equation here obtained differs from the one obtained by Darboux's method in the particular case $n = 3$, $m = 1$; this equation is (Theorie des Surfaces, vol. I, p. 22)

$$\frac{d\sigma}{dt} = - ip_{12}\sigma + \frac{p_{13} - ip_{23}}{2} + \frac{p_{13} + ip_{23}}{2}\sigma^2,$$

and has no analogue for $n > 3$. It was the failure of this method when applied to the general system that led me to search for a method which would be applicable in general, and it has been shown that this method is chiefly based upon orthogonal, or more explicitly, Euclidean transformations.

U. S. NAVAL ACADEMY, *December* 27, 1904.

On the Determination of the Properties of the Nodal Curve of a Unicursal Ruled Surface.

BY CHARLES H. SISAM.

The parametric equations of a unicursal ruled surface may be written:

$$(1)\qquad x_i = a_i(\mu) + v b_i(\mu), \qquad i = 1, 2, 3, 4.$$

wherein $a_i(\mu)$, $b_i(\mu)$ are polynomials in μ.

I shall suppose—it being no restriction on the surface—that there exists no value of μ which satisfies all the equations:

$$a_i(\mu)b_j(\mu) - a_j(\mu)b_i(\mu) = 0, \qquad i, j = 1, 2, 3, 4.$$

The generator along which μ has any particular value μ_1 will be referred to as the generator μ.

Any point of the nodal curve is determined by two pairs of values (μ, v) and (μ', v') of the parameters. Hence, if from the four equations,

$$(2)\qquad a_i(\mu) + v b_i(\mu) = \rho\,[a_i(\mu') + v' b_i(\mu')] \qquad i = 1, 2, 3, 4,$$

we eliminate ρ, μ', v', solve for v (or μ) and substitute in (1) we obtain the parametric equations of the nodal curve.

If, on the other hand, we eliminate v, ρ, v' from (2), we obtain:

$$(3)\qquad \begin{vmatrix} a_1(\mu) & a_1(\mu') & b_1(\mu) & b_1(\mu') \\ a_2(\mu) & a_2(\mu') & b_2(\mu) & b_2(\mu') \\ a_3(\mu) & a_3(\mu') & b_3(\mu) & b_3(\mu') \\ a_4(\mu) & a_4(\mu') & b_4(\mu) & b_4(\mu') \end{vmatrix} = 0.$$

This expression is divisible by $(\mu - \mu')^2$. The resulting equation is expressible in terms of $\mu + \mu'$ and $\mu\mu'$ only. Call it $F(\mu + \mu', \mu\mu') = 0$. Put $\mu + \mu' = \xi$ and $\mu\mu' = \eta$ and consider $F(\xi, \eta) = 0$ as the non-homogeneous equation of a curve whose current co-ordinates are (ξ, η). It then follows that:

Each irreducible component of $F(\xi, \eta) = 0$ *determines an irreducible component of the nodal curve of the surface. If the latter is double only, then the number* *of times it is cut by an arbitrary generator equals the order, and its genus equals the genus, of the component of $F = 0$ which determines it.*

For, let $\Phi(\xi, \mu) = 0$ be an irreducible component of $F = 0$. Let $p_{ik}(\mu) \equiv a_i(\mu) b_k(\mu) - b_i(\mu) a_k(\mu)$ be the line coordinates of the generator μ and $p'_{ik}(\mu')$ be those of the generator μ'. Then when $\Phi(\mu + \mu', \mu\mu') = 0$, the curve:

$$(4) \qquad \begin{cases} x_1 = (p_{12}p'_{13} - p_{13}p'_{12})(p_{34}p'_{42} - p_{42}p'_{34}) = \rho f_1(\mu + \mu', \mu\mu') \\ x_2 = (p_{42}p'_{23} - p_{23}p'_{42})(p_{34}p'_{14} - p_{14}p'_{34}) = \rho f_2(\mu + \mu', \mu\mu') \\ x_3 = (p_{34}p'_{13} - p_{13}p'_{34})(p_{34}p'_{42} - p_{42}p'_{34}) = \rho f_3(\mu + \mu', \mu\mu') \\ x_4 = (p_{34}p'_{14} - p_{14}p'_{34})(p_{34}p'_{42} - p_{42}p'_{34}) = \rho f_4(\mu + \mu', \mu\mu') \end{cases}$$

belongs to the double curve of the surface (1).

When the curve (4) is a double curve, only, on the surface, a one to one correspondence exists between its points and those of $\Phi(\xi, \eta) = 0$, for any two generators intersecting on (4) determine a single point (their intersection) on (4), and also a single point (the point $\xi = \mu + \mu'$, $\eta = \mu\mu'$) on $\Phi = 0$. Hence $\Phi = 0$ and the double curve (4) are of the same genus. Moreover, if in $\Phi(\mu + \mu', \mu\mu') = 0$ we give μ an arbitrary fixed value, we obtain m (= order of $\Phi(\xi, \eta) = 0$) values for μ', hence the generator μ meets m other generators on (4) and therefore meets (4) in m points.

Two distinct components of $F(\xi, \eta) = 0$ can not determine the same *double* curve on the surface (1). $F = 0$ may, however, have one, or more, other components identical with a given component $\Phi = 0$. The two sheets of the surface through the corresponding double curve (4) then touch, osculate, etc., according to the number of components of $F = 0$, identical with $\Phi = 0$. If Φ^i is a factor of F then the corresponding curve (4) counts i times as a component of the nodal curve.

A simple illustration (already obtained by another method by Snyder*) is that of a quintic scroll with an oscnodal conic. Writing the equations of the scroll in the form:

$$x_1 = 1\,; \quad x_2 = \mu^2 + v\,; \quad x_3 = \mu^3 - v\mu\,; \quad x_4 = v\mu^2,$$

the equation $F(\xi, \eta) = 0$ becomes $\xi^3 = 0$. The oscnodal conic is $4x_1 x_4 = x_2^2$, $x_3 = 0$. It counts for a nodal curve of order six.

A component of the nodal curve which is of multiplicity greater than two on the surface may, on the other hand, be determined by more than one component of the curve $F = 0$. Between the points of an i-fold curve on the surface and the totality of distinct components of $F = 0$ which determine it exists a $\left[1, \frac{i(i-1)}{2}\right]$ correspondence. These curves may be tacnodal, oscnodal, etc., under the same conditions as before. They exhibit, also, the following further singularity: Among the components of $F = 0$ which determine the same curve on the surface, some may enter to the first power only, others to the second, etc.; correspondingly, among the sheets of the surface passing through the curve some will simply intersect, others touch, etc. The equation $F = 0$, therefore, forms, as before, a criterion for the mode of intersection of the sheets of the surface along the curve.

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*Bulletin Am. Math. Soc., vol. 11, p. 182.

Consider, for example, a scroll having a fourfold conic along which the sheets of the surface touch in pairs. Such a surface may be obtained by setting up a $(1, 2)$ correspondence between the points of the conic and the planes through the tangent at the point, then joining the point of tangency to the points in which the corresponding planes cut another conic. To the former conic corresponds in $F = 0$, a quadratic factor and a linear factor squared. It counts for sixteen as a component of the nodal curve.

A more interesting illustration is that of a scroll having a fourfold quartic, along which two sheets of the surface touch, the remaining intersections being simple. Such a surface is the nonic:

$$\begin{aligned} x_1 &= \mu^4 + v(\mu^3 - 2\mu^4 + 2\mu^5) \\ x_2 &= \mu^2 + 2\mu^3 - 2\mu^4 + v\mu \\ x_3 &= \mu - 2\mu^2 + 2\mu^3 + v \\ x_4 &= 4 \end{aligned}$$

The equation $F = 0$ reduces to:

$$\xi^2 \left[\xi^5 - 2\xi^4 + (1 - 4\eta)\xi^3 + 6\eta\,\xi^2 + (4\eta^2 - \eta)\xi + 4\eta^2 - \eta\right] = 0.$$

The complete nodal curve is the cuspidal quartic:

$$x_1 = \lambda^4 \quad x_2 = \lambda^3 \quad x_3 = \lambda \quad x_4 = \quad -1.$$

It is equivalent to a nodal curve of order twenty-eight.

Let $\phi_1 = 0, \phi_2 = 0, \ldots, \phi_i = 0$ determine the same component of the nodal curve of the surface (1)(in general, $i = 1$). If in the product:—

$$\phi_1(\mu + \mu', \mu\mu') \cdot \phi_2(\mu + \mu', \mu\mu') \cdot \ldots \cdot \phi_i(\mu + \mu', \mu\mu') = 0$$

we put $\mu' = \mu$, the resulting equation determines the torsal generators of the surface which have their pinch-points on the curve considered.

When a ruled surface is developable, every point on its edge of regression is a pinch-point. The corresponding component of $F=0$ must, therefore, vanish identically for $\mu=\mu'$. That is, to the edge of regression of a developable always corresponds the conic, $\xi^2-4\eta=0$.

A theory similar to that formulated here for the double curve holds also for the double developable (i. e., the developable formed by the planes containing two generators), for to the point (ξ,η) on $F(\xi,\eta)=0$ corresponds the plane of the intersecting generators, $\frac{\xi\pm\sqrt{\xi^2-4\eta}}{2}$ and conversely. Thus, in the case of the quintic scroll with an oscnodal conic mentioned above, we have seen that $F=0$ reduces to $\xi^3=0$. It readily follows that the double developable consists of the tangents to a quadric cone counted thrice. This surface has, in fact, the interesting property of being self-dual without belonging to a linear complex.*

When the surface has a rectilinear directrix, we may take the directrix to be $x_1=x_2=0$, and write the equations of the surface in the form:

$$x_1=\vartheta(\mu)a(\mu);\quad x_2=\vartheta(\mu)b(\mu);\, x_3=c(\mu)+vc'(\mu);\quad x_4=d(\mu)+vd'(\mu),$$

wherein $a(\mu)$ and $b(\mu)$ are relatively prime.

Equation (3) now reduces to:

$$\vartheta(\mu)\,\vartheta(\mu')\begin{vmatrix} a(\mu) & a(\mu') \\ b(\mu) & b(\mu') \end{vmatrix}\;\begin{vmatrix} c'(\mu) & c'(\mu') \\ d'(\mu) & d'(\mu') \end{vmatrix}=0.$$

$\vartheta(\mu)=0$ or $\vartheta(\mu')=0$ determines the parameters of the generators coincident with the directrix. The equation

$$\frac{1}{\mu-\mu'}\begin{vmatrix} c'(\mu) & c'(\mu') \\ d'(\mu) & d'(\mu') \end{vmatrix}=0,$$

is satisfied by the parameters of the generators that intersect on $x_1=x_2=0$. It determines the properties of the double developable other than $x_1+\lambda x_2=0$. The properties of the nodal curve—other than the directrix itself—are determined by the consideration of the expression:

$$\frac{1}{\mu-\mu'}\begin{vmatrix} a(\mu) & a(\mu') \\ b(\mu) & b(\mu') \end{vmatrix}=0.$$

The determination of the double curve and of the double developable of such a surface is, therefore, closely related to the theory of binary forms.

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* See Wilczynski, Math. Annalen, vol. 58, p. 249; also Bull. Amer. Math. Soc., vol. 11, p. 8. Sisam, Bull. Amer. Math. Soc., vol. 10, p. 440.

U. S. NAVAL ACADEMY, *September*, 1905.

Certain Surfaces with Plane or Spherical Lines of Curvature.

By L. P. Eisenhart.

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§1. INTRODUCTION.

Elsewhere we have considered the surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces; we have called them the *A-surfaces.*[*] We have found that they admit of transformations into new A-surfaces in a way which is a generalization of the Bäcklund transformations of pseudospherical surfaces. In the present paper we wish to study the A-surfaces with plane or spherical lines of curvature in one system and to apply the earlier results to these particular surfaces.

After a brief recall in §2 of the needed results of our former study, we proceed in §3 to the discussion of A-surfaces with plane lines of curvature in one or both systems. We show that the moulure surfaces are the only A-surfaces with the lines in both systems plane, and then find the character of the curves on the sphere which represent plane lines in one system on A-surfaces.

In §4 we consider the A-surfaces of the Joachimstal type, first in terms of the parameters used in §§2 and 3, and then in terms of a set of parameters so chosen as to enable us to give the cartesian coordinates of the surface in a simple form. We find that among all the A-surfaces with the same spherical representation and having plane lines of curvature in one system there is an infinity of the Joachimstal type, the pseudospherical surface being one of the latter.

In §5 after noting that for all A-surfaces with plane lines of curvature in one system the planes of the lines envelope a cylinder, which reduces to a line for the surfaces of Joachimstal, we make use of an investigation by Raffy, *Sur les surfaces a lignes de courbure planes dont les plans enveloppent un cylindre* (*Annales*

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[*] Amer. Journ., Vol. 27, pp. 113-172.

de L'Ecole Normale, 1901), and are able to find the expressions for the cartesian coordinates of all these surfaces.

When the generalization of the Bäcklund transformation of pseudospherical surfaces is a generalization of the transformation of Bianchi of the latter, we call it the *complementary* transformation (see §2). In §6 we find that every complementary transform of a moulure surface has plane lines of curvature in one system; that every surface with plane lines of curvature in one system has two at most of such transforms; that this is true of no other A-surfaces.

In §7 we find the conditions to be satisfied in order that the curves $v=$ const. be spherical and certain particular cases are discussed, including surfaces of Joachimstal which are of this class also. In §8 it is given to find the character of an A-surface in order that it may admit of a complementary transform with spherical lines of curvature in one system. And, finally, it is shown that there are an infinity of A-surfaces, each of which admits of an infinity of transforms with spherical lines of curvature in one system.

§2. *General Formulae.*

The parameters of the lines of curvature upon a pseudospherical surface of unit curvature can be so chosen that the linear element of the surface can be written

$$ds^2=\cos^2\omega du^2+\sin^2\omega dv^2,* \tag{1}$$

where ω is a solution of the equation

$$\frac{\partial^2\omega}{\partial u^2}-\frac{\partial^2\omega}{\partial v^2}=\sin\omega\cos\omega; \tag{2}$$

and every solution of this equation determines a pseudospherical surface. The linear element of the spherical representation is

$$d\sigma^2=\sin^2\omega du^2+\cos^2\omega dv^2.* \tag{3}$$

This evidently is the expression for the linear element of the spherical representation of the lines of curvature of certain A-surfaces, and each solution ω of equation (2) gives rise to a whole group of these. For if the linear element of such a surface be written

$$ds^2=A^2du^2+C^2dv^2, \tag{4}$$

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* Darboux, Leçons, t. III, p. 377.

the coefficients A and C must satisfy the conditions

$$-\frac{1}{C}\frac{\partial A}{\partial v}=\frac{\partial \omega}{\partial v}, \quad \frac{1}{A}\frac{\partial C}{\partial u}=\frac{\partial \omega}{\partial u}, \tag{5}$$

and every set of functions A and C satisfying these equations determines an A-surface. We have shown* that the solution of the system (5) requires the integration of a Laplace equation, from which one gets an idea of the number of A-surfaces in a *group* with the same spherical representation.

Consider the system of equations

$$\left\{\begin{aligned} &\frac{\partial \theta}{\partial u}+\frac{\partial \omega}{\partial v}=\cos\omega\sin\theta,\\ &\frac{\partial \theta}{\partial v}+\frac{\partial \omega}{\partial u}=-\sin\omega\cos\theta. \end{aligned}\right. \tag{6}$$

The integration of this system is reducible to the solution of a Riccati equation; consequently the general integral involves an arbitrary constant. Given a particular integral θ_1; in the tangent planes to an A-surface with the spherical representation (3), draw the lines through the points of contact, making the angle θ_1 with the lines of curvature $v=$ const. We have shown† that the planes perpendicular to the tangent planes and meeting them in these lines envelope a new A-surface S_1 with the lines of curvature parametric, and for which the linear element of the spherical representation is

$$d\sigma_1^2=\sin^2\theta_1 du^2+\cos^2\theta_1 dv^2; \tag{7}$$

and the coefficients of the linear element of S_1 have the expression

$$\left\{\begin{aligned} A_1&=\frac{\partial A}{\partial u}+C\frac{\partial \omega}{\partial u}+\cos\theta_1(A\cos\omega+C\sin\omega),\\ C_1&=\frac{\partial C}{\partial v}-A\frac{\partial \omega}{\partial v}+\sin\theta_1(A\cos\omega+C\sin\omega). \end{aligned}\right. \tag{8}$$

We call this transformation *complementary;* when the given surface is pseudospherical it is the transformation discovered by Bianchi and called after him by Darboux.

§3. *A-Surfaces with Plane Lines of Curvature.*

In order that the lines of curvature $v=$ const. on a surface be plane it is necessary and sufficient that the spherical representation of these lines be small

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[*] Amer. Journ., l. c. p. 118.

[†] Amer. Journ., l. c. p. 129.

circles, and consequently that the lines have constant geodesic curvature. The general expression for the geodesic curvature of the lines $v =$ const. is[*]

$$\frac{1}{\rho_v} = -\frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v},$$

where E and G have their customary significance. Hence, if the given surface is an A-surface with the spherical representation (3), the condition that the curves $v =$ const. on the sphere be circles is

$$\frac{1}{\sin \omega} \frac{\partial \omega}{\partial v} = V, \tag{9}$$

where V denotes a function of v alone. In a similar manner the necessary and sufficient condition that the lines $u =$ const. be plane is

$$\frac{1}{\cos \omega} \frac{\partial \omega}{\partial u} = U, \tag{10}$$

where U denotes a function of u alone. If these conditions are to be satisfied simultaneously, we must have

$$\frac{\partial}{\partial u}(\sin \omega \, . \, V) = \frac{\partial}{\partial v}(\cos \omega U),$$

which reduces by means of (9) and (10) to

$$U . V = 0.$$

Assume that $V = 0$, then it follows from (9) that ω is a function of u alone. We have shown[†] that this is the necessary and sufficient condition that the corresponding A-surfaces be moulure surfaces. Hence *moulure surfaces are the only A-surfaces with plane lines of curvature in both systems.*

We limit ourselves now to the case where only the lines $v =$ const. are plane. Equation (9) must be satisfied, which may be written in either of the forms

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin \omega} \frac{\partial \omega}{\partial v}\right) = 0, \quad \frac{\partial}{\partial v}\left(\frac{1}{\sin \omega} \frac{\partial \omega}{\partial u}\right) = 0. \tag{11}$$

Write these equations in the form

$$\frac{\partial \omega}{\partial u} = U \sin \omega, \quad \frac{\partial \omega}{\partial v} = V \sin \omega \tag{12}$$

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[*] Bianchi, Lezioni I, 181.

[†] Amer. Journ., l. c. p. 122.

and substitute in equation (2); by means of (12) the result may be reduced to

$$\cos\omega = \frac{V' - U'}{U^2 - (V^2 + 1)}. \tag{13}$$

When this value of ω is substituted in the first of (12), it becomes

$$UU'' - \frac{U''}{U} - U'^2 - (U^2 - 1)^2 - V^2\frac{U''}{U} + 2(U^2 - 1)V^2 + V'^2 - V^4 = 0; \tag{14}$$

in like manner the second of (12) becomes

$$VV'' + \frac{V''}{V} - V'^2 - (U^2 - 1)^2 - U^2\frac{V''}{V} + 2(U^2 - 1)V^2 + U'^2 - V^4 = 0. \tag{15}$$

Differentiate (14) with respect to v; this gives

$$V'\left[V\frac{U''}{U} - 2(U^2 - 1)V + 2V^3 - V''\right] = 0. \tag{16}$$

If V' is zero, ω as given by (13) is a function of u alone, in which case the corresponding surfaces are moulure surfaces, and consequently the lines of curvature are plane in both systems. Excluding this case, equation (16) may be written

$$\frac{U''}{U} - 2(U^2 - 1) + 2V^2 - \frac{V''}{V} = 0,$$

which may be replaced by the two equations

$$\frac{U''}{U} - 2(U^2 - 1) = a, \quad \frac{V''}{V} - 2V^2 = a, \tag{17}$$

where a denotes a constant. From the first of these we have

$$U'^2 = U^4 + (a - 2)U^2 + b \tag{18}$$

where b is a constant. When this value is substituted in (14), one gets

$$V'^2 = V^4 + aV^2 + (a + b - 1). \tag{19}$$

This value of V is seen to be consistent with the second of (17). And these values of U and V determined by these equations satisfy equation (15). Hence every solution of equations (18) and (19) gives a configuration upon the sphere which serves for the spherical representation of the lines of curvature of A-surfaces with the lines $v =$ const. plane. As a and b are arbitrary constants, there is a double infinity of these systems and they are determined by two quadratures, as is seen from the form of (18) and (19).

§4. *A-Surfaces of Joachimstal.*

Surfaces with plane lines of curvature in one system, such that all the planes pass through a line L, are called surfaces of Joachimstal. We inquire whether any of these surfaces are A-surfaces.

The necessary and sufficient condition that a surface S with plane lines of curvature in one family be a surface of Joachimstal is that the lines of curvature in the other system lie on spheres which cut the surface orthogonally; in this case the centres of the spheres lie on the line L*. If a curve C on S is the intersection of S and a sphere, which cuts the surface orthogonally, the radii of the sphere to the points of C form the developable circumscribing S along C, and the centre of the sphere, since it is the vertex of the cone, is the centre of the geodesic curvature of C, by the theorem of Beltrami;† evidently this curvature is constant. Conversely, Brioschi has shown that when a line of curvature has constant geodesic curvature, it is the intersection of the surface and a sphere cutting the surface orthogonally.‡ Hence our problem reduces to the determination of the A-surfaces with the lines $v=$ const. plane, and with the lines of curvature $u=$ const. of constant geodesic curvature.

In consequence of (5) the geodesic curvature of the lines $u=$ const. is given by§

$$\frac{1}{\rho_u}=-\frac{1}{AC}\frac{\partial C}{\partial u}=-\frac{1}{C}\frac{\partial \omega}{\partial u},$$

hence for these lines to be of constant geodesic curvature we must have

$$\frac{\partial}{\partial v}\left(\frac{1}{C}\frac{\partial \omega}{\partial u}\right)=0.$$

At the same time equations (11) must obtain; consequently we must have

$$C=U_1\sin\omega, \tag{20}$$

where U_1 is any function whatever of u alone. From the second of equations (5) we find that

$$A=U_1\cos\omega+\frac{\sin\omega\cdot U_1'}{\frac{\partial \omega}{\partial u}}, \tag{21}$$

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* Bianchi, Lezioni, Vol. II, p. 269.

† Bianchi, Ib., Vol. I, p. 277.

‡ Bianchi, Lezioni, Vol. II, p. 296.

§ Bianchi, Lezioni, I, p. 181.

where the accent denotes differentiation with respect to u. In consequence of equations (11) these expressions for A and C satisfy the first of equations (5). Hence we have the theorem:

For each value of ω given by (13) *there is a family of A-surfaces of the Joachimstal type, depending upon an arbitrary function.*

Their linear element is of the form

$$ds^2 = \left(U_1 \cos\omega + \frac{U_1'}{U}\right)^2 du^2 + U_1^2 \sin^2\omega \, dv^2, \tag{22}$$

where U_1 is arbitrary and U is determined by (18). When U_1 is equal to unity, (22) reduces to the form (1). We remarked that for every pseudospherical surface whose curvature is equal to -1 the linear element can be given this form; and when the curvature is another constant the linear element is reducible to the form (1) multiplied by the constant. Hence pseudospherical surfaces with plane lines of curvature in one system are surfaces of Joachimstal. This result was obtained first by Enneper,* and on this account pseudospherical surfaces of this kind are called *surfaces of Enneper.*

From another point of view we shall be able to find the coordinates of these surfaces in finite form. For this we remark that Raffy† has shown that any surface of Joachimstal can be defined by the equations

$$\frac{x}{\cos v} = \frac{y}{\sin v} = \frac{1}{U' \cosh(U+V)}, \quad z = u - \frac{\sinh(U+V)}{U' \cosh(U+V)}, \tag{23'}$$

where U and V are functions of u and v respectively, and the curves $v = \text{const.}$ are plane lines of curvature and $u = \text{const.}$ their orthogonal trajectories. One finds for the linear element of the spherical representation of the surface, as thus defined,

$$d\sigma^2 = \frac{\sin^2\mu}{\cosh^2(U+V)} dU^2 + \left[\frac{\sinh(U+V)}{\cosh(U+V)} - \mu'\right]^2 dv^2,$$

where μ is a function of v alone defined by

$$\cot\mu = V'.$$

If we put $U = u_1$, or write it $u = U_1$ where U_1 is such a function of u_1 that when u in U is replaced by U_1 it becomes u_1, the equations of the surface become

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* Gött. Nachrichten, 1868, p. 258.

† Sur les surfaces à lignes de courbure planes, etc., Ann. L'Ecole Normale, ser. III, Vol. 18, p. 357 (1901).

(on removing the subscript from u_1)

$$(23)\qquad \frac{x}{\cos v}=\frac{y}{\sin v}=\frac{U_1'}{\cosh(u+V)},\quad z=U_1-U_1'\frac{\sinh(u+V)}{\cosh(u+V)},$$

where the prime denotes differentiation with respect to u. Now the expression for the linear element on the sphere is

$$(24)\qquad d\sigma^2=\frac{\sin^2\mu}{\cosh^2(u+V)}du^2+\left[\frac{\sinh(u+V)}{\cosh(u+V)}-\mu'\right]^2dv^2.$$

From the form of this expression it follows that V determines the configuration on the sphere, and the character of U_1 has no effect.

Darboux* has given the following general method of generating a surface of Joachimstal: In a plane which passes through a line L one draws any family of circles with centres on L, and also their orthogonal trajectories; if the plane is made to turn about L according to any given law whatever, as one passes from one trajectory to the other, the locus of the trajectories is a surface of Joachimstal. From this statement and the preceding remark concerning (24) it follows that U_1 determines the array of circles, and V the motion of the plane. For our problem U_1 is arbitrary, and V must be determined so that (24) is the representation of the lines of curvature of a pseudospherical surface.

In order to do this we seek the forms of the functions U and V so that equations (23′) shall define a surface of Enneper. When a surface is defined by (23′), its principal radii of curvature are given by

$$(25)\qquad \begin{cases} \rho_1=\dfrac{\sqrt{1+V'^2}}{U'}\left[\sinh(U+V)+\dfrac{U''}{U'^2}\cosh(U+V)\right], \\[2ex] \rho_2=\dfrac{-\sqrt{1+V'^2}}{U'\left[\sinh(U+V)+\dfrac{V''}{1+V'^2}\cosh(U+V)\right]}. \end{cases}$$

From these expressions we find that the necessary and sufficient condition that the curvature of the surface be -1 is

$$(26)\qquad \sinh(U+V)\left[1-\frac{U'^2}{1+V'^2}\right]+\cosh(U+V)\left[\frac{U''}{U'^2}-\frac{V''U'^2}{(1+V'^2)^2}\right]=0.$$

From (25) it is seen that $U'\neq 0$. We consider first the case where $V'=0$; from the form of the foregoing expressions it is seen that in all generality we may take $V=0$, but then equations (23′) define surfaces of revolution which are an evident solution of the problem, and will be excluded from the further discussion. With this understanding we put

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* Leçons, Vol. 1, p. 117.

$$(27)\qquad U_2 = \frac{1}{U'^2}, \quad V_2 = \frac{1}{1 + V'^2}$$

and take U and V for independent variables, designating differentiation with respect to them by accents. Equation (26) reduces to

$$(28)\qquad 2\tanh(U+V)(U_2 - V_2) = U_2' - V_2'.$$

Differentiate this equation with respect to V, and multiply by $\cosh^2(U+V)$; this gives

$$2(U_2 - V_2) - V_2'\sinh 2(U+V) + V_2''\cosh^2(U+V) = 0.$$

If this equation be differentiated with respect to V, the result reduces to

$$V_2''' = 4V_2',$$

from which it follows that

$$(29)\qquad V_2 = a\cosh 2V + b\sinh 2V + c,$$

where a, b, c are constants. From the form of (28) it follows that in like manner

$$(29')\qquad U_2 = a_1\cosh 2U + b_1\sinh 2U + c_1.$$

When these values are substituted in (27) we find

$$a_1 = -a, \quad b_1 = b, \quad c_1 = c.$$

From (27) and (29) it follows that the function V determining a system of lines on the sphere which serve for the representation of the lines of curvature of an A-surface of Joachimstal is given by the quadrature

$$(30)\qquad \int\sqrt{\frac{a\cosh 2V + b\sinh 2V + c}{-a\cosh 2V - b\sinh 2V - (c-1)}}\,dV = v.$$

When the expression for V from this equation is substituted in (23′) or (23) they define all the A-surfaces of Joachimstal. When the parameter u has been chosen so that the equations take the form (23), the form of U_1 for the corresponding surface of Enneper is given by the quadrature

$$U_1 = \int\sqrt{c - a\cosh 2u + b\sinh 2u}\,.\,du,$$

in consequence of (27) and (29′).

The form of (18) is not altered if u and V be replaced by $u + x$ and $V - x$ respectively, where x is a constant which can be so chosen as to remove the term $\cosh 2V$ or $\sinh 2V$ from (29). Hence only two of the constants a, b, c are arbitrary, so that there is on the sphere only a double infinity of the curves sought, as was found before.

§5. *A-surfaces with Plane Lines of Curvature in One System.*

We proceed now to the determination of the A-surfaces with plane lines of curvature in one system which are not necessarily surfaces of Joachimstal. We remark that the planes of these lines of curvature are parallel to the corresponding planes of the lines of curvature of the pseudospherical surface with the same spherical representation as the given surface. But, as we have seen, the planes for the pseudospherical surface pass through a line L, and consequently for the given surface they either pass through a line or envelope a cylinder whose elements are parallel to L. We have considered the former in the preceding section and now will turn to the latter. For this the recent investigations by Raffy* will be of great service. He remarks that if S is a surface with plane lines of curvature $v = \text{const.}$, and S_1 is the locus of the centres of normal curvature of these lines, the geodesics $v = \text{const.}$ on S_1 are the curves of contact of circumscribing cylinders, and the angle between the parametric lines on S_1 is a function of v. When, in particular, the planes are parallel to a line L the circumscribing cylinders are parallel to a plane normal to L and conversely. Taking the z-axis for the line L, Raffy† finds, for the surfaces for which the circumscribing cylinders are parallel to the xy plane, expressions which can be put in the form

$$
(31)\quad \begin{cases}
x_1 = (U_1' \cos v + U_1'' \sin v \cot \mu) \cosh (u + V) \\
\qquad\qquad - (U_1'' \cos v + U_1' \sin v \cot \mu) \sinh (u + V) + \psi \cos v - \psi' \sin v, \\
y_1 = (U_1' \sin v - U_1'' \cos v \cot \mu) \cosh (u + V) \\
\qquad\qquad - (U_1'' \sin v - U_1' \cos v \cot \mu) \sinh (u + V) + \psi \sin v + \psi' \cos v, \\
z_1 = U_1 - U_1'',
\end{cases}
$$

where U_1 is an arbitrary function of u, ψ an arbitrary function of v; μ denotes the angle between the parametric lines on S_1 and $\cot \mu = V'$, the function V depending only upon v and, as we have seen, determining the spherical representation for S.

From (31) one finds the direction-cosines of the tangents to the curves $v = \text{const.}$ on S_1; as these are the normals to S, we denote them by X, Y, Z, and find for their values

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*Sur les surfaces a lignes de courbure planes, etc. Ann. L'Ecole Normale, ser. III, Vol. 18, p. 341.

†Ib., p. 352.

$$(32)\qquad \begin{cases} X = \cos v \sin \mu \tanh(u + V) - \sin v \cos \mu, \\ Y = \sin v \sin \mu \tanh(u + V) + \cos v \cos \mu, \\ Z = \dfrac{\sin \mu}{\cosh(u + V)}. \end{cases}$$

The surface S is defined by

$$(33)\qquad x = x_1 - \lambda X, \quad y = y_1 - \lambda Y, \quad z = z_1 - \lambda Z,$$

where λ is determined by the condition

$$\Sigma dx_1 . X - d\lambda = 0,$$

or

$$(34)\qquad \frac{\partial \lambda}{\partial u} = \Sigma X \frac{\partial x_1}{\partial u}, \quad \frac{\partial \lambda}{\partial v} = \Sigma X \frac{\partial x_1}{\partial v}.$$

The first of these may be written

$$\frac{\partial \lambda}{\partial u} = \sqrt{E_1} = \frac{(U_1' - U_1''') \cosh(u + V)}{\sin \mu},$$

from which we get

$$(35)\qquad \lambda = \frac{U_1' \sinh(u + V) - U_1'' \cosh(u + V)}{\sin \mu} + \chi(v),$$

where χ is a function of v alone. The second of (34) reduces, in consequence of the definition of μ, to

$$\frac{\partial \lambda}{\partial v} = \cos \mu \sqrt{G_1} = \frac{\cos \mu}{\sin^2 \mu} [(U_1' + \mu' U_1'') \cosh(u + V) - (U_1' \mu' + U_1'') \sinh(u + V)] \\ + [\psi(v) + \psi''(v)] \cos \mu.$$

When the value of λ from (35) is substituted in this equation, it is found that the function $\chi(v)$ is determined by

$$(36)\qquad \chi'(v) = [\psi(v) + \psi''(v)] \cos \mu.$$

From (33), (32), (31) and (35) we have for the coordinates of any A-surface with the lines $v =$ const. plane

$$(37)\qquad \begin{cases} x = \left[\dfrac{U_1' - \chi \sin \mu \sinh(u + V)}{\cosh(u + V)} + \psi \cos v\right] \cos v + [\chi \cos v - (\psi \cos v)'] \sin v, \\ y = \left[\dfrac{U_1' - \chi \sin \mu \sinh(u + V)}{\cosh(u + V)} + \psi \cos v\right] \sin v - [\chi \cos v - (\psi \cos v)'] \cos v, \\ z = U_1 - \dfrac{U_1' \sinh(u + V) + \chi \sin \mu}{\cosh(u + V)}, \end{cases}$$

where V is determined by the quadrature (30), U_1 and $\psi(v)$ are arbitrary and $\chi(v)$ is given by (36). For $\psi = 0$ and consequently $\chi = 0$, formulæ (37) reduce to (23), that is, $\psi = 0$ is the necessary and sufficient condition that they be surfaces of Joachimstal. From (37) we derive the following expressions for the coefficients of the linear element of S

$$(38)\qquad \left\{\begin{aligned} E &= \left[\frac{\sinh(u+V).U_1' + \chi\sin\mu}{U_1'\cosh(u+V)} - \frac{U_1''}{U_1'}\right]^2, \\ G &= \left[\frac{U_1'}{\sin\mu\cosh(u+V)} - \chi\{\tanh(u+V) - \mu'\} + \chi'\tan\mu\right]^2. \end{aligned}\right.$$

In particular, when $a = b = 0$ in (30) it reduces to

$$V = \sqrt{\frac{1-c}{c}}\,v + \text{const.}$$

Hence if the parameters are real, $0 < c < 1$. Now

$$\cot\mu = 1 + V'^2 = \frac{1}{c},$$

hence the parametric lines on S_1 cut one another under constant angle.

§6. *Complementary Transformations of Certain A-Surfaces.*

Consider an A-surface S with the spherical representation of its lines of curvature given by (3). Let S_1 be a complementary transform corresponding to a particular integral θ of the system (6). Further we assume that the lines of curvature $v =$ const. on S_1 are plane; for this we must have (11)

$$\frac{\partial}{\partial v}\left(\frac{1}{\sin\theta}\frac{\partial\theta}{\partial u}\right) = 0.$$

By means of (6) this equation may be replaced by

$$(39)\qquad \sin\theta\frac{\partial^2\omega}{\partial v^2} + \cos\theta\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} + \sin\omega\frac{\partial\omega}{\partial v} = 0.$$

This equation is satisfied identically when $\frac{\partial\omega}{\partial v} = 0$, that is, when S is a moulure surface. Hence:

The complementary transforms of a moulure surface have plane lines of curvature in one system.

From (39) it is seen that, if S is not a moulure surface, two at most of its complementary transforms have plane lines of curvature in one system. The angle θ giving these transforms is obtained from (39); but this function must satisfy (6) also. Instead of solving (39) and substituting in (6), we differentiate the former with respect to v and take account of the latter. This gives

$$\sin\theta\left[\frac{\partial^3\omega}{\partial v^3}+\left(\frac{\partial\omega}{\partial u}\right)^2\frac{\partial\omega}{\partial v}-\sin^2\omega\,\frac{\partial\omega}{\partial v}\right]+\cos\theta\,\frac{\partial\omega}{\partial v}\,\frac{\partial^2\omega}{\partial u\partial v}+\cos\omega\left(\frac{\partial\omega}{\partial v}\right)^2=0.$$

The values of θ given by this equation must be the same as in (39), hence we must have

$$(40)\qquad \frac{\dfrac{\partial^3\omega}{\partial v^3}+\left(\dfrac{\partial\omega}{\partial u}\right)^2\dfrac{\partial\omega}{\partial v}-\sin^2\omega\,\dfrac{\partial\omega}{\partial v}}{\dfrac{\partial^2\omega}{\partial v^2}}=\frac{\dfrac{\partial^2\omega}{\partial u\partial v}}{\dfrac{\partial\omega}{\partial u}}=\frac{\cos\omega\,\dfrac{\partial\omega}{\partial v}}{\sin\omega}.$$

From the last two we have

$$(41)\qquad \frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\,\frac{\partial\omega}{\partial v}\right)=0,$$

that is, the lines of curvature $v=$ const. on S are plane. In consequence of (2), the first two ratios of (40) are equal when (41) is satisfied. In like manner we differentiate (39) with respect to u and take account of (6); this gives

$$\sin\theta\left[\frac{\partial^3\omega}{\partial u\partial v^2}+\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2\right]+\cos\theta\,\frac{\partial\omega}{\partial u}\,\frac{\partial^2\omega}{\partial u\partial v}+\sin\omega\,\frac{\partial^2\omega}{\partial u\partial v}=0,$$

so that ω must satisfy

$$\frac{\partial\omega}{\partial v}\,\frac{\partial^3\omega}{\partial u\partial v^2}+\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3-\frac{\partial^2\omega}{\partial u\partial v}\,\frac{\partial^2\omega}{\partial v^2}=0.$$

But this equation also is a consequence of (41). Hence:

If the lines $v=$ const. on an A-surface are plane, the two values of θ given by (39) *determine complementary transformations, such that the lines $v=$ const. on the transforms are plane, and on all the other transforms the lines $v=$ const. are skew.*

An exception to this theorem arises for the case where the two values of θ, given by (39), coincide. The necessary and sufficient condition for this is

$$(42)\qquad \left(\frac{\partial^2\omega}{\partial v^2}\right)^2+\left[\left(\frac{\partial\omega}{\partial u}\right)^2-\sin^2\omega\right]\left(\frac{\partial\omega}{\partial v}\right)^2=0,$$

and the single value is given by

$$(43)\qquad \cos\theta=-\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2\sin\omega.$$

Suppose we have given a function ω, satisfying (42), and we denote by S a surface with the spherical representation of its lines of curvature determined by ω. Let S_0 be a surface of which S is the complementary transform by means of ω, and let its spherical representation be given by

$$d\sigma_0^2 = \sin^2\phi du^2 + \cos^2\phi dv^2;$$

then ϕ is a particular integral of the system

$$(44) \qquad \frac{\partial\omega}{\partial u} + \frac{\partial\phi}{\partial v} = \sin\omega\cos\phi, \quad \frac{\partial\omega}{\partial v} + \frac{\partial\phi}{\partial u} = -\cos\omega\sin\phi.$$

In consequence of the above results we know that the lines $v = \text{const.}$ on S_0 are plane, so that ϕ must satisfy

$$(45) \qquad \frac{\partial}{\partial v}\left(\frac{1}{\sin\phi}\frac{\partial\phi}{\partial u}\right) = 0.$$

By means of (45) and (44) equation (42) may be replaced by

$$\operatorname{cosec}^2\phi \, . \left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial\phi}{\partial u} + \cos\omega\sin\phi\right)^2 = 0,$$

which reduces to

$$\frac{\partial\phi}{\partial v} \cdot \frac{\partial\omega}{\partial v} = 0.$$

If ω were a function of u alone, all the transforms of S would have plane lines of curvature $v = \text{const.}$ Hence the above equation gives $\phi = \phi(u)$, and consequently S_0 is a moulure surface. We have then the theorem:

If a surface S is a complementary transform of a moulure surface, it admits only one complementary transformation giving rise to a surface with plane lines of curvature in one system; the determining function θ of the transformation is given by (43); *for all other A-surfaces with plane lines of curvature there are two such transformations.*

§7. *A-Surfaces with Spherical Lines of Curvature.*

Enneper* has shown that the necessary and sufficient condition that the lines of curvature $v = \text{const.}$ on a surface be spherical is that the following relation obtains

$$A = R\cos\sigma\frac{A}{\rho_1} + R\sin\sigma\frac{\rho_2}{C}\frac{\partial}{\partial v}\frac{A}{\rho_1},$$

where R is the radius of the sphere and σ the angle under which the latter cuts the surface; both of these quantities are functions of v alone.

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* Gött. Nachrichten, 1868, p. 421.

For an A-surface

$$A = -\rho_1 \sin\omega, \quad C = \rho_2 \cos\omega,$$

so that for these surfaces the equation of condition is

$$A = \alpha \sin\omega + \beta \frac{\partial\omega}{\partial v}, \tag{46}$$

where α and β are functions of v alone.

Consider first the case where the surface is pseudospherical; equation (46) becomes

$$\cot\omega = \alpha + \frac{\beta}{\sin\omega}\frac{\partial\omega}{\partial v}.$$

Differentiate with respect to u; this gives

$$\frac{1}{\sin^2\omega}\frac{\partial\omega}{\partial u} + \beta\frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right) = 0.$$

Differentiate again with respect to u and eliminate β. This gives the following condition to be satisfied by ω:

$$\begin{vmatrix} \dfrac{1}{\sin^2\omega}\dfrac{\partial\omega}{\partial u} & \dfrac{\partial}{\partial u}\left(\dfrac{1}{\sin\omega}\dfrac{\partial\omega}{\partial v}\right) \\ \dfrac{\partial}{\partial u}\left(\dfrac{1}{\sin^2\omega}\dfrac{\partial\omega}{\partial u}\right) & \dfrac{\partial^2}{\partial u^2}\left(\dfrac{1}{\sin\omega}\dfrac{\partial\omega}{\partial v}\right) \end{vmatrix} = 0, \tag{47}$$

which reduces to

$$\frac{\partial\omega}{\partial u}\frac{\partial^3\omega}{\partial u^2\partial v} - \frac{\partial^2\omega}{\partial u^2}\frac{\partial^2\omega}{\partial u\partial v} + \left(\frac{\partial\omega}{\partial u}\right)^3\frac{\partial\omega}{\partial v} = 0. \tag{48}$$

It is evident that equation (47) is satisfied when

$$\frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right) = 0;$$

hence when equation (48) is satisfied the curves $v =$ const. on the pseudospherical surface are *spherical or plane.* In like manner when ω satisfies the equation

$$\frac{\partial\omega}{\partial v}\frac{\partial^3\omega}{\partial u\partial v^2} - \frac{\partial^2\omega}{\partial v^2}\frac{\partial^2\omega}{\partial u\partial v} + \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3 = 0, \tag{49}$$

the lines of curvature $u =$ const. on the corresponding pseudospherical surface are spherical, unless

$$\frac{\partial}{\partial u}\left(\frac{1}{\cos\omega}\frac{\partial\omega}{\partial v}\right) = 0, \tag{50}$$

in which case they are plane. These results have been found by Bianchi.*

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*Aggiunte alla Memoire "Sopra i sistemi tripli, etc." Annali di Mat., Vol. 14, p. 116 (1887).

We pass to the consideration of A-surfaces in general with one family of spherical lines of curvature. From (46) and the first of equations (5) we have

$$C = -\left(\frac{\alpha' \sin\omega + \beta \dfrac{\partial^2\omega}{\partial v^2}}{\dfrac{\partial\omega}{\partial v}} + \beta' + \alpha\cos\omega\right), \tag{51}$$

where the accent denotes differentiation with respect to v. When this expression for C is substituted in the second of equations (5), the latter becomes

$$\alpha'\left[\cos\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} - \sin\omega\frac{\partial^2\omega}{\partial u\partial v}\right] + \beta\left[\frac{\partial\omega}{\partial v}\frac{\partial^3\omega}{\partial u\partial v^2} - \frac{\partial^2\omega}{\partial v^2}\frac{\partial^2\omega}{\partial u\partial v} + \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3\right] = 0. \tag{52}$$

If the expression in the first parenthesis is zero, the same is true of the second; as this is the condition that the lines $v =$ const. be plane, it may be neglected.

Another solution of this equation is obtained by taking α' equal to zero, and also the expression in the second parenthesis. The vanishing of the latter is the condition (49) that the lines $u =$ const. be plane or spherical on the pseudospherical surface, having the same spherical representation as the given surface; hereafter we shall say that this pseudospherical surface is *associated* with the given surface. Two cases are to be considered according as α is equal to or different from zero; in the latter case it may be taken equal to unity.

1° $\alpha = 0$. From the definition of α and β it follows that in this case σ is a right-angle and β is the expression for the radius of the sphere; hence when β is constant the spheres upon which the curves $v =$ const. lie are spherical. The linear element of the surface is

$$ds^2 = \beta^2\left(\frac{\partial\omega}{\partial v}\right)^2 du^2 + \left(\beta\frac{\partial^2\omega}{\partial v^2} + \beta'\frac{\partial\omega}{\partial v}\right)^2 \frac{dv^2}{\left(\dfrac{\partial\omega}{\partial v}\right)^2}, \tag{53}$$

where β is an arbitrary function of v alone.

2° $\alpha = 1$. The projection of the radius of the sphere on the normal to the surface is constant. All the spheres have the same radius when β is constant, in which case the spheres and the surface cut under constant angle and only in this case. The linear element of these surfaces is

$$ds^2 = \left(\sin\omega + \beta\frac{\partial\omega}{\partial v}\right)^2 du^2 + \left(\cos\omega + \beta' + \beta\frac{\dfrac{\partial^2\omega}{\partial v^2}}{\dfrac{\partial\omega}{\partial v}}\right)dv^2. \tag{54}$$

Gathering together these results we have the theorem:

Given a pseudospherical surface with lines of curvature $u=$ const. plane or spherical; among all the A-surfaces with which it is associated there are two families, each containing an infinity of surfaces whose lines of curvature $v=$ const. are spherical.

When the lines $u=$ const. on the pseudospherical surface are plane, they are plane for all the A-surfaces associated with it. Now

$$\frac{\partial\omega}{\partial v}=\cos\omega\,.\,V,$$

where V is a function of v alone. If we substitute this value in (53), it becomes

$$ds^2=V_1^2\cos^2\omega du^2+\left(\frac{V_1'}{V}-V_1\sin\omega\right)^2dv^2, \tag{55}$$

where we have put $V_1=\beta V$, so that V_1 is arbitrary and V is determined in a manner similar to U in §3. If (55) is compared with (22), it will be seen that the former defines the linear element of the A-surfaces of Joachimstal with $u=$ const. plane lines of curvature. In a similar manner we have from (54) in this special case

$$ds^2=(V_1\cos\omega+\sin\omega)^2du^2+\left(\cos\omega-V_1\sin\omega+\frac{V_1'}{V}\right)^2dv^2. \tag{56}$$

From this we have the theorem:

In addition to the family of surfaces of Joachimstal associated with a pseudospherical surface of Enneper there is another infinite family of A-surfaces with plane lines of curvature in one system and spherical lines in the other. Their linear element is of the form (56).

Excluding these particular solutions of (52), we differentiate it with respect to u and eliminate α' and β, with the result

$$\begin{vmatrix} \cos\omega\dfrac{\partial\omega}{\partial u}\dfrac{\partial\omega}{\partial v}-\sin\omega\dfrac{\partial^2\omega}{\partial u\partial v}\,, & \dfrac{\partial\omega}{\partial v}\dfrac{\partial^3\omega}{\partial u\partial v^3}-\dfrac{\partial^2\omega}{\partial v^2}\dfrac{\partial^2\omega}{\partial u\partial v}+\dfrac{\partial\omega}{\partial u}\left(\dfrac{\partial\omega}{\partial v}\right)^3 \\ \dfrac{\partial}{\partial u}\left[\cos\omega\dfrac{\partial\omega}{\partial u}\dfrac{\partial\omega}{\partial v}-\sin\omega\dfrac{\partial^2\omega}{\partial u\partial v}\right], & \dfrac{\partial}{\partial u}\left[\dfrac{\partial\omega}{\partial v}\dfrac{\partial^3\omega}{\partial u\partial v^2}-\dfrac{\partial^2\omega}{\partial v^2}\dfrac{\partial^2\omega}{\partial u\partial v}+\dfrac{\partial\omega}{\partial u}\left(\dfrac{\partial\omega}{\partial v}\right)^3\right] \end{vmatrix}=0. \tag{57}$$

When a solution of this equation is found and substituted in (52), α' can be chosen arbitrarily, and β is entirely determined; the coefficients of the linear element are given by (46) and (51). Hence the theorem:

Given a solution ω of equation (57) which satisfies (2); this function serves to determine the spherical representation of the lines of curvature of an infinity of A-surfaces on which the lines of curvature $v=$ const. are spherical.

When ω satisfies equation (49), it is a solution of (57); consequently we have the theorem:

When the lines of curvature $u = const.$ on a pseudospherical surface are spherical, there is an infinity of A-surfaces associated with it whose lines $v = const.$ are spherical; and the fundamental coefficients are given without quadrature.

Moreover, when ω satisfies (50) it is a solution not only of (49), but of (48); this can be shown by differentiating (50) with respect to u. Hence the theorem:

When the lines of curvature in one system on an A-surface are plane, the lines of curvature in the other system are spherical.

§8. *Complementary Transforms with Spherical Lines of Curvature.*

Bianchi has considered* the problem of finding the pseudospherical surfaces whose complementary transforms have spherical lines of curvature in one system. We shall recall his investigation briefly before proceeding to the discussion of the same problem for A-surfaces in general.

Denote by Σ the complementary transform of the pseudospherical surface S and write the linear element of the spherical representation of the former in the form

$$d\sigma^2 = \sin^2\theta du^2 + \cos^2\theta dv^2, \tag{58}$$

recalling that θ and ω are bound by the relations

$$\frac{\partial\theta}{\partial u} + \frac{\partial\omega}{\partial v} = \sin\theta\cos\omega, \quad \frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u} = -\cos\theta\sin\omega. \tag{59}$$

For the lines $v = \text{const.}$ on Σ to be spherical θ must satisfy an equation of the form (47), which by means of (59) can be written

$$A\cos\theta + B\sin\theta + C = 0,$$

where we have put

$$A = \sin\omega\left(\frac{\partial\omega}{\partial u}\right)^2\frac{\partial\omega}{\partial v} + \cos\omega\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial u\partial v},$$

$$B = \cos\omega\frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2 + \cos\omega\frac{\partial^3\omega}{\partial u\partial v^2} + \sin\omega\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial v^2},$$

$$C = \sin^2\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} - \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3 + \sin\omega\cos\omega\frac{\partial^2\omega}{\partial u\partial v} - \frac{\partial\omega}{\partial v}\frac{\partial^3\omega}{\partial u\partial v^2} + \frac{\partial^2\omega}{\partial v^2}\frac{\partial^2\omega}{\partial u\partial v}.$$

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* l. c. p. 118.

From the above it follows that for more than two of the transforms to have the desired property, it is necessary that A, B and C be zero.

Putting A equal to zero, we find that either $\frac{\partial\omega}{\partial u}$ must be zero or ω be a solution of equation (50). In the former case S is a surface of revolution with the meridians $u = \text{const.}$, and we have noted the latter is the condition that the curves $u = \text{const.}$ on S be plane. These values make B and C vanish so that we have the theorem of Bianchi:

For all the complementary transforms of a pseudospherical surface S to have spherical lines of curvature $v = \text{const.}$ it is necessary and sufficient that the lines of curvature $u = \text{const.}$ on S be plane, that is, S must be a surface of Enneper. Bianchi has shown that the transforms are surfaces of Enneper only in case S is a surface of revolution.

We proceed now to the general problem of finding the A-surfaces whose complementary transforms have spherical lines of curvature in one system, say $v = \text{const.}$ Let the spherical representation of these lines be given by (58) with the relations (59). Now θ must satisfy an equation of the form (57), which by means of (59) can be reduced to

$$\begin{vmatrix} l\cos\theta + m\sin\theta + n & a\cos\theta + b\sin\theta + c \\ \frac{\partial}{\partial u}(l\cos\theta + m\sin\theta + n) & \frac{\partial}{\partial u}(a\cos\theta + b\sin\theta + c) \end{vmatrix} = 0, \tag{60}$$

where we have put, for the sake of brevity,

$$\left\{\begin{aligned} l &= \frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v}, \quad m = \frac{\partial^2\omega}{\partial v^2}, \quad n = \sin\omega\frac{\partial\omega}{\partial v}, \\ a &= \sin\omega\left(\frac{\partial\omega}{\partial u}\right)^2\frac{\partial\omega}{\partial v} + \frac{\partial^3\omega}{\partial u^2\partial v}\sin\omega - \cos\omega\frac{\partial\omega}{\partial v}\frac{\partial^2\omega}{\partial u^2}, \\ b &= \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^2\cos\omega - \sin\omega\frac{\partial\omega}{\partial v}\frac{\partial^2\omega}{\partial u\partial v}, \\ c &= \frac{\partial\omega}{\partial u}\frac{\partial^3\omega}{\partial u^2\partial v} + \frac{\partial\omega}{\partial v}\left(\frac{\partial\omega}{\partial u}\right)^3 - \frac{\partial^2\omega}{\partial u^2}\frac{\partial^2\omega}{\partial u\partial v} - \cos^2\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} + \sin\omega\cos\omega\frac{\partial^2\omega}{\partial u\partial v}. \end{aligned}\right. \tag{61}$$

From the discussion of equation (57) it follows that equation (60) is satisfied when the lines $v = \text{const.}$ on the transforms are plane, that is when

$$\frac{\partial}{\partial v}\left(\frac{1}{\sin\theta}\frac{\partial\theta}{\partial u}\right) = 0.$$

But we have seen that, unless the given surface is a moulure-surface, only two of the transforms will have the curves $v = \text{const.}$ plane. Hence when equation

(60) is satisfied for more than two values of θ, the lines $v = \text{const.}$ on the corresponding transforms are spherical.

We return now to equation (60) and find that when the determinant is developed the equation can be written

$$A + B\cos\theta + C\sin\theta + D\sin\theta\cos\theta + E\sin^2\theta = 0,$$

where we have put

$$(62)\quad \begin{cases} A = nc' - n'c + a'l - al' + (am - bl)\dfrac{\partial\omega}{\partial v}, \\ B = na' - n'a + c'l - cl' + (cm - bn)\dfrac{\partial\omega}{\partial v}, \\ C = nb' - n'b + c'm - cm' + (an - cl)\dfrac{\partial\omega}{\partial v} + (bl - am)\cos\omega, \\ D = lb' - l'b + a'm - am' + (bn - cm)\cos\omega, \\ E = mb' - m'b + al' - a'l + (cl - an)\cos\omega, \end{cases}$$

the primes denoting differentiation with respect to u. From the form of the above equation it is clear that for more than four of the transforms to have spherical lines of curvature, it is necessary that A, B, C, D, E vanish, for we are dealing with real surfaces.

If A and B are equated to zero and the resulting equations are solved for a' and c', one finds

$$(63)\quad \begin{aligned} &\left(\frac{\partial\omega}{\partial v}\right)^2\left[\left(\frac{\partial\omega}{\partial u}\right)^2 - \sin^2\omega\right]\left\{a' - \left[\left(\frac{\partial\omega}{\partial u}\right)^2 + \frac{\dfrac{\partial^3\omega}{\partial u^2\partial v}}{\dfrac{\partial\omega}{\partial v}}\right]\left(\sin\omega\frac{\partial^2\omega}{\partial u\partial v} + \cos\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v}\right)\right. \\ &\quad + 2\cos\omega\frac{\partial^2\omega}{\partial u\partial v}\frac{\partial^2\omega}{\partial u^2} + \cos^3\omega\frac{\partial\omega}{\partial u}\frac{\partial\omega}{\partial v} - \sin\omega\cos^2\omega\frac{\partial^2\omega}{\partial u\partial v} - \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3\cos\omega \\ &\quad \left. + \sin\omega\left(\frac{\partial\omega}{\partial v}\right)^2\frac{\partial^2\omega}{\partial u\partial v}\right\} = 0, \end{aligned}$$

$$(63)\quad \begin{aligned} &\left(\frac{\partial\omega}{\partial v}\right)^2\left[\left(\frac{\partial\omega}{\partial u}\right)^2 - \sin^2\omega\right]\left\{c' - \left[\left(\frac{\partial\omega}{\partial u}\right)^2 + \frac{\dfrac{\partial^3\omega}{\partial u^2\partial v}}{\dfrac{\partial\omega}{\partial v}}\right]\left(\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial u\partial v} + \sin\omega\cos\omega\frac{\partial\omega}{\partial v}\right)\right. \\ &\quad - \sin\omega\cos\omega\frac{\left(\dfrac{\partial^2\omega}{\partial u\partial v}\right)^2}{\dfrac{\partial\omega}{\partial v}} + \cos^2\omega\frac{\partial\omega}{\partial v}\frac{\partial^2\omega}{\partial u^2} + \frac{\dfrac{\partial^2\omega}{\partial u^2}\left(\dfrac{\partial^2\omega}{\partial u\partial v}\right)^2}{\dfrac{\partial\omega}{\partial v}} \\ &\quad \left. + \cos^2\omega\frac{\partial\omega}{\partial u}\frac{\partial^2\omega}{\partial u\partial v}\right\} = 0. \end{aligned}$$

These equations are satisfied by $\frac{\partial\omega}{\partial v}$ equal to zero, and from the expressions for a, b, c it is clear that equation (60) also is satisfied by this value. But we have seen that for this case the curves $v =$ const. are plane for the transforms, and consequently may be neglected.

Equations (63) are satisfied also when

$$\left(\frac{\partial\omega}{\partial u}\right)^2 - \sin^2\omega = 0,$$

from which we get

$$\tan\frac{\omega}{2} = u\,.\,V, \tag{64}$$

where V is a function of v alone. From the above equation and (2) we find that $\frac{\partial^2\omega}{\partial v^2}$ is zero, and for ω given by (64) to satisfy this condition V must be a constant; hence this case leads back to the former.

We consider finally the case where the expressions in the brackets in (63) are zero. If the values of a' and c' given in this manner be substituted in the expressions for C and D and the latter be equated to zero, we get

$$\tag{65} \begin{gathered} a\left[m'\frac{\partial\omega}{\partial v} - m\frac{\partial^2\omega}{\partial u\partial v} + \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3\right] = 0, \\ b\left[m'\frac{\partial\omega}{\partial v} - m\frac{\partial^2\omega}{\partial u\partial v} + \frac{\partial\omega}{\partial u}\left(\frac{\partial\omega}{\partial v}\right)^3\right] = 0. \end{gathered}$$

From (61) it follows that if b is equal to zero, we have $\frac{\partial\omega}{\partial v}$ zero, which has been considered, or ω must satisfy (11). When the latter is satisfied, both a and c vanish and equation (60) is satisfied. Hence when the lines $v =$ const. on the surface are plane and consequently $u =$ const. are spherical, the lines $v =$ const. on the transforms, or surfaces with the same spherical representation, are spherical.

There is one further case to be considered, namely, when the expression in the parenthesis of (65) is zero, which reduces readily to (49). When this equation holds, the values of a' and c' obtained from (61) are found to satisfy (63) and make E equal to zero. Hence:

The necessary and sufficient condition that for each solution θ of equations (59) *there is an infinity of surfaces with the lines $v =$ const. spherical is that ω be a solution of equations* (2) *and* (49).

Suppose now that we have a function ω satisfying these conditions. Among all the surfaces with the corresponding representation of the lines of curvature there must be an infinity which admit by means of a solution of equations (59) transforms upon which the lines $v = \text{const.}$ are spherical. We inquire whether there are any of them for which the transforms by all the functions θ, given by (59), have spherical lines of curvature $v = \text{const.}$

Denote by A_1 and C_1 the coefficients of the linear element for one of the transforms S_1. Now it must be possible for A_1 and C_1 to assume forms of expression similar to (46) and (51). Hence from (8) the necessary and sufficient condition that all the transforms of the same surface have spherical lines of curvature $v = \text{const.}$ is that the following equations be satisfied:

$$(66)\quad \begin{cases} \dfrac{\partial A}{\partial u} + C\dfrac{\partial \omega}{\partial u} + \lambda \cos\theta = \alpha \sin\theta + \beta\dfrac{\partial\theta}{\partial v}, \\[2ex] \dfrac{\partial C}{\partial v} - A\dfrac{\partial\omega}{\partial v} + \lambda\sin\theta = -\dfrac{\dfrac{\partial\alpha}{\partial v}\sin\theta + \beta\dfrac{\partial^2\theta}{\partial v^2} + \left(\dfrac{\partial\beta}{\partial v} + \alpha\sin\theta\right)\dfrac{\partial\theta}{\partial v}}{\dfrac{\partial\theta}{\partial v}} \end{cases}$$

for every value of θ satisfying (59), and the corresponding values of α and β given by

$$(67)\quad \frac{\partial\alpha}{\partial v}\left[\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v} - \sin\theta\frac{\partial^2\theta}{\partial u\partial v}\right] + \beta\left[\frac{\partial\theta}{\partial v}\frac{\partial^3\theta}{\partial u\partial v^2} - \frac{\partial^2\theta}{\partial v^2}\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial v}\right)^3\right] = 0,$$

where we have put

$$(68)\qquad \lambda = A\cos\omega + C\sin\omega.$$

The general solution θ of (59) involves a parameter, say t, which in general will appear in α and β, but not in A, C and ω. If then equations (66) are differentiated with respect to t, they become

$$(69)\quad \begin{aligned} \lambda\sin\theta\frac{\partial\theta}{\partial t} &= -\left[\alpha\cos\theta\frac{\partial\theta}{\partial t} + \sin\theta\frac{\partial\alpha}{\partial t} + \frac{\partial\theta}{\partial v}\frac{\partial\beta}{\partial t} + \beta\frac{\partial^2\theta}{\partial v\partial t}\right], \\ \lambda\cos\theta\frac{\partial\theta}{\partial t} &= -\left[\frac{\sin\theta\dfrac{\partial^2\alpha}{\partial v\partial t} + \cos\theta\dfrac{\partial\alpha}{\partial v}\dfrac{\partial\theta}{\partial t} + \dfrac{\partial\beta}{\partial t}\dfrac{\partial^2\theta}{\partial v^2} + \beta\dfrac{\partial^3\theta}{\partial t\partial v^2}}{\dfrac{\partial\theta}{\partial v}} + \frac{\partial^2\beta}{\partial v\partial t}\right. \\ &\qquad \left. + \frac{\partial\alpha}{\partial t}\cos\theta - \alpha\sin\theta\frac{\partial\theta}{\partial t} - \frac{\dfrac{\partial\alpha}{\partial v}\sin\theta + \beta\dfrac{\partial^2\theta}{\partial v^2}}{\left(\dfrac{\partial\theta}{\partial v}\right)^2}\right]. \end{aligned}$$

Eliminating λ, we are brought to the equation

$$\begin{aligned}(70)\qquad & -\alpha+\frac{\sin^2\theta}{\frac{\partial\theta}{\partial t}\frac{\partial\theta}{\partial v}}\frac{\partial^2\alpha}{\partial v\partial t}+\frac{\sin\theta}{\left(\frac{\partial\theta}{\partial v}\right)^2}\left(\cos\theta\frac{\partial\theta}{\partial v}-\sin^2\theta\sin\omega\right)\frac{\partial\alpha}{\partial v}\\ & +\frac{1}{\frac{\partial\theta}{\partial t}\frac{\partial\theta}{\partial v}}\left[\sin\theta\frac{\partial^2\theta}{\partial v^2}-\cos\theta\left(\frac{\partial\theta}{\partial v}\right)^2\right]\frac{\partial\beta}{\partial t}+\frac{\sin\theta}{\frac{\partial\theta}{\partial t}}\frac{\partial^2\beta}{\partial v\partial t}\\ & +\frac{\sin\theta}{\left(\frac{\partial\theta}{\partial v}\right)^2}\left[\left(\sin\theta-\cos\omega\frac{\partial\theta}{\partial u}\right)\frac{\partial\theta}{\partial v}-\sin\omega\frac{\partial^2\theta}{\partial v^2}\right]\beta=0.\end{aligned}$$

Differentiate this equation with respect to u and subtract the two equations; eliminate $\frac{\partial^2\alpha}{\partial v\partial t}$ from this result by means of its value as given by differentiating (67); this gives

$$\begin{aligned}&\frac{\partial\alpha}{\partial v}-\frac{\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}-\sin\theta\frac{\partial^2\theta}{\partial u\partial v}}{\frac{\partial\theta}{\partial t}\frac{\partial\omega}{\partial v}}\frac{\partial^2\beta}{\partial v\partial t}+\left[\sin\theta\frac{\partial^3\theta}{\partial u\partial v^2}+\sin\theta\frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial v}\right)^2-\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial^2\theta}{\partial v^2}\right.\\ &\left.-\frac{\cos\theta\,.\left(\cos\theta\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}-\sin\theta\frac{\partial^2\theta}{\partial u\partial v}\right)\left(\frac{\partial^2\theta}{\partial u\partial v}-\cos\omega\cos\theta\frac{\partial\theta}{\partial v}-\frac{\partial\theta}{\partial v}\right)}{\cos\theta\frac{\partial\omega}{\partial v}+\sin\theta}\right]\frac{\frac{\partial\beta}{\partial t}}{\frac{\partial\theta}{\partial t}\frac{\partial\omega}{\partial v}}=0.\end{aligned}$$

When this equation is differentiated with respect to u, the result reduces in consequence of (49) to

$$R\frac{\partial\beta}{\partial t}=0,$$

where R is a function which does not vanish for the above value of ω. Hence β is independent of t and the preceding equation shows that α does not involve v. Then the expression in the second parenthesis of (67) is zero*; but the latter is reducible to the form $a\cos\theta+b\sin\theta+c$, where a, b, c have the values (61), and for this to vanish for all values of θ the coefficients must be zero, from which it follows that ω must satisfy

$$(11)\qquad \frac{\partial}{\partial u}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial v}\right)=0.$$

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* β cannot be zero, for in such case we should have from (46) and (51)

$$A=a\sin\omega,\quad C=-a\cos\omega,$$

which are the coefficients for the sphere, when a is constant.

Now equation (70) reduces to $\alpha = 0$, and from equations (69) it is seen that

$$\lambda = -\beta \sin \omega .$$

By means of (5) equation (68) can now be written

$$\cos \omega \frac{\partial C}{\partial u} + C \sin \omega \frac{\partial \omega}{\partial u} = -\beta \sin \omega \frac{\partial \omega}{\partial u},$$

from which we get by integration

$$C = -\beta + \gamma \cos \omega ,$$

γ being a function of v alone. Then for A by (5) we have

$$A = -\gamma \sin \omega .$$

These values for A and C satisfy the first of equations (66), and the second reduces to

$$\gamma' = -\frac{\beta}{\sin \omega}\frac{\partial \omega}{\partial v}.$$

From (11) it follows that $\frac{1}{\sin \omega}\frac{\partial \omega}{\partial v}$ is a function of v alone, hence all the conditions will be satisfied when γ is given by

$$\gamma = -\int \frac{\beta}{\sin \omega}\frac{\partial \omega}{\partial v} dv + \text{const.} \tag{71}$$

Hence we have the following theorem:

The necessary and sufficient condition that the lines of curvature $v = \text{const.}$ be spherical for all the transforms of an A-surface, which is not pseudospherical, is that the lines $v = \text{const.}$ be plane on the original surface, and the latter be one of the infinity of surfaces with the linear element

$$ds^2 = \gamma^2 \sin^2 \omega du^2 + (\gamma \cos \omega - \beta)^2 dv^2,$$

where β is arbitrary and γ is given by (71). The linear element of a transform is

$$ds_1^2 = \beta^2\left(\frac{\partial \theta}{\partial v}\right)^2 du^2 + \left(\beta \frac{\partial^2 \theta}{\partial v^2} + \frac{\partial \beta}{\partial v}\frac{\partial \theta}{\partial v}\right)^2 \frac{dv^2}{\left(\frac{\partial \theta}{\partial v}\right)^2}.$$

Since not more than two of these surfaces can have the lines $v = \text{const.}$ plane all the transforms of these surfaces cannot have spherical lines of curvature $v = \text{const.}$

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The Motion of a Solid in Infinite Liquid.

A. G. Greenhill.

The present paper is a sequel to the one with the same title in Vol. XX, of the *American Journal of Mathematics*, 1897; it carries on the investigation for the external shape of the body shown by Clebsch in *Mathematische Annalen* III, p. 238, to lead to an elliptic function solution of the same character as before in the simple shape of revolution considered by Kirchhoff in his *Vorlesungen.*

Contrary to anticipation derived from a study of Halphen's treatment of the same problem in his *Fonctions elliptiques* II. Chap. 4, of which this article may be considered a commentary, the extension from Kirchhoff's shape to the more general form discussed by Clebsch does not introduce a complication essentially greater. Moreover the extra constant at disposal enables us to construct an algebraical case of the motion with greater ease, much as the discussion of the symmetrical top is in many respects simpler in its analysis than the more restricted case of the Spherical Pendulum.

The elliptic-function solution of the motion under no force of a solid in infinite liquid is then the object of this memoir; for the more general case expressible by the double-theta hyperelliptic function, initiated by H. Weber, the *Fortschritte der Mathematik* must be consulted for references to the discussion of various authors, such as H. Weber, F. Kötter, R. Liouville, Caspary, Jukovsky, Liaponoff, and others.

1. We employ the notation of Halphen at the outset, with the occasional introduction of a symbol from the paper in the *American Journal*; and now with Clebsch's extended form of the kinetic energy for helicoidal symmetry

$$(1)\qquad \left\{\begin{aligned} T = {} & \tfrac{1}{2}p(x_1^2 + x_2^2) + \tfrac{1}{2}p'x_3^2 \\ & + q(x_1y_1 + x_2y_2) + q'x_3y_3 \\ & + \tfrac{1}{2}r(y_1^2 + y_2^2) + \tfrac{1}{2}r'y_3^2, \end{aligned}\right.$$

and the dynamical equations

$$(2)\qquad \frac{dx_1}{dt} = x_3\frac{\partial T}{\partial y_2} - x_2\frac{\partial T}{\partial y_3}, \ldots\ldots, \ldots\ldots,$$

$$(3)\qquad \frac{dy_1}{dt} = -x_3\frac{\partial T}{\partial x_2} + x_2\frac{\partial T}{\partial x_3} - y_3\frac{\partial T}{\partial y_2} + y_2\frac{\partial T}{\partial y_3}, \ldots\ldots, \ldots\ldots$$

we arrive immediately at the three integrals

$$(4)\qquad 2T = \text{constant} = l,$$

$$(5)\qquad x_1^2 + x_2^2 + x_3^2 = \text{constant} = m,$$

$$(6)\qquad x_1y_1 + x_2y_2 + x_3y_3 = \text{constant} = n,$$

suppose; and in addition

$$(7)\qquad \frac{dy_3}{dt} = 0, \text{ so that } y_3 \text{ is constant.}$$

Denoting the component linear and angular velocity with respect to axes OA, OB, OC, fixed in the body by

$$(8)\qquad U,\ V,\ W,\ P,\ Q,\ R,$$

as in Halphen, but a change to capital letters from the notation in the *American Journal*,

$$(9)\qquad U = \frac{\partial T}{\partial x_1} = px_1 + qy_1,$$

$$(10)\qquad V = \frac{\partial T}{\partial x_2} = px_2 + qy_2,$$

$$(11)\qquad W = \frac{\partial T}{\partial x_3} = p'x_3 + q'y_3.$$

Introducing Euler's unsymmetrical angles θ, ϕ, ψ defining the position of OA, OB, OC with respect to axes OX, OY, OZ having fixed direction in space, on the system employed in Klein-Sommerfeld, *Theorie des Kreisels*, p. 19, in accordance with figure 1, and the scheme

	A	B	C
X	$\cos\phi\cos\psi - \cos\vartheta\sin\phi\sin\psi$	$-\sin\phi\cos\psi - \cos\vartheta\cos\phi\sin\psi$	$\sin\vartheta\sin\psi$
Y	$\cos\phi\sin\psi + \cos\vartheta\sin\phi\cos\psi$	$-\sin\phi\sin\psi + \cos\vartheta\cos\phi\cos\psi$	$-\sin\vartheta\cos\psi$
Z	$\sin\vartheta\sin\phi$	$\sin\vartheta\cos\phi$	$\cos\vartheta$

$$(12)\qquad \frac{\partial T}{\partial y_1} = qx_1 + ry_1 = P = \cos\phi\frac{d\vartheta}{dt} + \sin\vartheta\sin\phi\frac{d\psi}{dt},$$

$$(13)\qquad \frac{\partial T}{\partial y_2} = qx_2 + ry_2 = Q = -\sin\phi\frac{d\vartheta}{dt} + \sin\vartheta\cos\phi\frac{d\psi}{dt},$$

so that

$$(14)\qquad q(x_1 + x_2 i) + r(y_1 + y_2 i) = P + Qi = \left(\frac{d\vartheta}{dt} + i\sin\vartheta\frac{d\psi}{dt}\right)e^{-\phi i};$$

$$(15)\qquad \frac{\partial T}{\partial y_3} = q'x_3 + r'y_3 = R = \frac{d\phi}{dt} + \cos\vartheta\frac{d\psi}{dt}.$$

From the third equation of system (2)

$$(16)\qquad \begin{cases} \dfrac{dx_3}{dt} = x_2\dfrac{\partial T}{\partial y_1} - x_1\dfrac{\partial T}{\partial y_2} \\ \quad = x_2(qx_1 + ry_1) - x_1(qx_2 + ry_2) \\ \quad = r(x_2y_1 - x_1y_2), \end{cases}$$

and from (6), distinguishing this n by an accent,

$$(17)\qquad x_1y_1 + x_2y_2 = n' - x_3y_3,$$

so that

$$(18)\qquad \frac{1}{r^2}\left(\frac{dx_3}{dt}\right)^2 = (x_2y_1 - x_1y_2)^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2) - (x_1y_1 + x_2y_2)^2$$

Again from (5) and (4)

$$x_1^2 + x_2^2 = m - x_3^2 \tag{19}$$

$$\left\{\begin{aligned} r(y_1^2 + y_2^2) &= l - p(m - x_3^2) - p'x_3^2 \\ &\quad - 2q(n' - x_3y_3) - 2q'x_3y_3 - r'y_3^2 \\ &= (p - p')x_3^2 + 2(q - q')x_3y_3 - mp - 2n'q - r'y_3^2 + l \\ &= (p - p')(x_3^2 + hx_3 - m_1) \end{aligned}\right. \tag{20}$$

on putting

$$(p - p')m_1 = mp + 2n'q + r'y_3^2 - l, \tag{21}$$

$$(p - p')h = 2(q - q')y_3; \tag{22}$$

and now

$$\left(\frac{dx_3}{dt}\right)^2 = r(p' - p)(x_3^2 - m)(x_3^2 + hx_3 - m_1) - r^2(x_3y_3 - n')^2, \tag{23}$$

a quartic in x_3, as in Halphen's (13), so that x_3 is an elliptic function of the time t.

Writing F^2 for m, as in the *American Journal* (*Am. J. M.*), then from Halphen's equation (45), F. E. II, p. 159,

$$x_1 = F\cos AZ = F\sin\vartheta\sin\phi, \tag{24}$$

$$x_2 = F\cos BZ = F\sin\vartheta\cos\phi; \tag{25}$$

and introducing Klein's α, β, γ, δ defined by

$$\begin{aligned} \alpha &= \cos\tfrac{1}{2}\vartheta e^{\frac{1}{2}(\phi+\psi)i}, & \beta &= i\sin\tfrac{1}{2}\vartheta e^{\frac{1}{2}(-\phi+\psi)i}, \\ \gamma &= i\sin\tfrac{1}{2}\vartheta e^{\frac{1}{2}(\phi-\psi)i}, & \delta &= \cos\tfrac{1}{2}\vartheta e^{\frac{1}{2}(-\phi-\psi)i}, \end{aligned} \tag{26}$$

$$x_1 + x_2 i = F(\cos AZ + i\cos BZ) = iF\sin\vartheta e^{-\phi i} = 2F\beta\delta, \tag{27}$$

$$x_3 = F\cos CZ = F\cos\vartheta = F(\alpha\delta + \beta\gamma); \tag{28}$$

thus expressing the constancy in magnitude and direction of the resultant momentum F of the system, taken as acting in the direction OZ.

Writing z for $\cos\vartheta$, equation (23) becomes

$$\left(\frac{dz}{dt}\right)^2 = F^2 r(p' - p)(z^2 - 1)\left(z^2 + \frac{h}{F}z - \frac{m_1}{F^2}\right) - \left(\frac{Fry_3z - n'r}{F}\right)^2, \tag{29}$$

and putting

$$F^2 r(p' \sim p) = n^2,\ F^2 r(p' - p) = an^2, \tag{30}$$

so that

(31) $a = +1$ when $p' - p$ is positive, as for prolate bodies,

(32) $a = -1$ .... negative, ... oblate ....

then

(33) $$\left(\frac{dz}{dt}\right)^2 = n^2 Z,$$

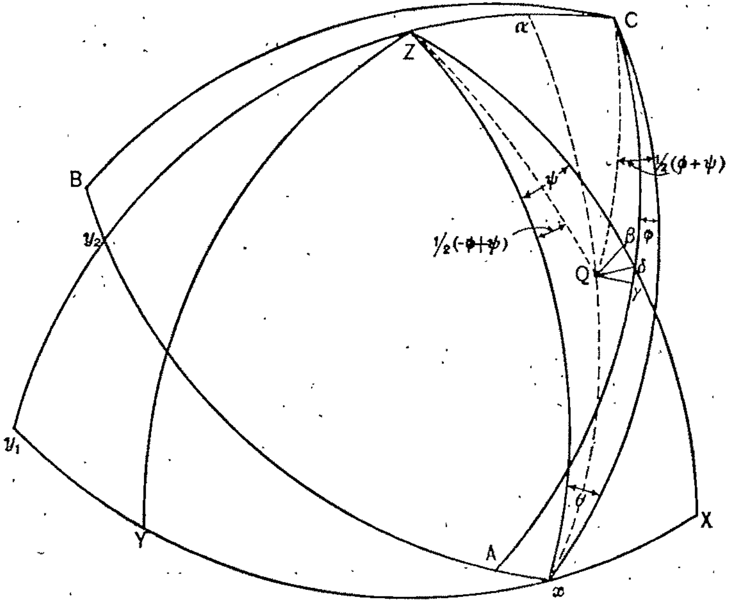
where

(34) $$Z = a(z^2 - 1)\left(z^3 + \frac{h}{F}z - \frac{m_1}{F^2}\right) - \left(\frac{Fry_3 z - n'r}{Fn}\right)^2,$$

and

(35) $$nt + \varepsilon = \int \frac{dz}{\sqrt{Z}},$$

an elliptic integral of the I kind.



2. The system of axes (X, Y, Z) can be brought into coincidence with the system (A, B, C) either by the successive rotation through the Eulerian angles ψ, ϑ, ϕ, about the axes OZ, Ox, OC in succession, as shown in figure 1; or else by a single rotation about an axis OQ, through an angle ω suppose.

Then if a, b, c denote the angles which OQ makes with the axes (X, Y, Z) or (A, B, C), the quaternion versor Q which performs the displacement of (X, Y, Z) into (A, B, C) is given by

(1) $$Q = Ai + Bj + Ck + D,$$

where

(2) $$A = \sin\tfrac{1}{2}\omega\cos a,\ B = \sin\tfrac{1}{2}\omega\cos b,\ C = \sin\tfrac{1}{2}\omega\cos c,\ D = \cos\tfrac{1}{2}\omega.$$

Now in the figure, Q lies on αx, the bisector of the angle $ZxC = \vartheta$, so that $QZx = QCx$; and drawing the perpendiculars from Q, $Q\beta$ on ZX, $Q\gamma$ on CA, ZX and CA intersecting in δ, the spherical triangles $QZ\beta$, $QC\gamma$ are congruent, and $QZ\beta = QC\gamma$. Thus

(3) $\quad QZx = QCx =$ half sum of XZx, $ACx = \frac{1}{2}(\phi + \psi)$,

(4) $\quad QZ\delta = QC\delta =$ half difference, ... $= \frac{1}{2}(-\phi + \psi)$;

and thence

(5) $\quad D = \cos\frac{1}{2}\omega \quad = \cos\alpha QZ \quad = \cos\frac{1}{2}\vartheta\cos\frac{1}{2}(\phi + \psi)$,

(6) $\quad C = \sin\frac{1}{2}\omega\cos c = \cos\alpha Z\cos\alpha ZQ = \cos\frac{1}{2}\vartheta\sin\frac{1}{2}(\phi + \psi)$.

Similarly

(7) $\quad A = \sin\frac{1}{2}\vartheta\cos\frac{1}{2}(-\phi + \psi),\ B = \sin\frac{1}{2}\vartheta\sin\frac{1}{2}(-\phi + \psi)$;

and thus

(8) $$A = \frac{\beta + \gamma}{2i}, \quad B = \frac{-\beta + \gamma}{2}, \quad C = \frac{\alpha - \delta}{2i}, \quad D = \frac{\alpha + \delta}{2},$$

(9) $$\alpha = D + Ci, \quad \beta = Ai - B, \quad \gamma = Ai + B, \quad \delta = D - Ci,$$

(Klein-Sommerfeld *Kreisel theorie*, p. 21).

In Routh's *Rigid Dynamics* the sequence of displacement is ψ about OZ and then ϑ about Oy_1, leading to a result of appearance slightly different.

3. The relations

(1) $$x_1 + x_2 i = \frac{r(U + Vi) - q(P + Qi)}{pr - q^2}, \quad x_3 = \frac{r'W - q'R}{p'r' - q'^2},$$

(2) $$y_1 + y_2 i = \frac{-q(U + Vi) + p(P + Qi)}{pr - q^2}, \quad y_3 = \frac{-q'W + p'R}{p'r' - q'^2},$$

obtainable from (9), (10), (11), (14), (15), §1, give x_1, x_2, x_3, y_1, y_2, y_3 in terms of the linear and angular components of velocity, and hence the kinetic energy expressed as a quadratic function of U, V, W, P, Q, R becomes

(3) $$\begin{cases} T = \frac{1}{2}\dfrac{r(U^2 + V^2) - 2q(UP + VQ) + p(P^2 + Q^2)}{pr' - q^2} \\ \quad + \frac{1}{2}\dfrac{r'W^2 - 2q'WR + p'R^2}{p'r' - q'^2} \end{cases}$$

as in the *American Journal of Mathematics* where q and q' are zero; and now the six dynamical equations of Kirchhoff become

$$(4)\qquad \frac{d}{dt}\frac{\partial T}{\partial U} - R\frac{\partial T}{\partial V} + Q\frac{\partial T}{\partial W} = 0, \ \ldots\ldots, \ \ldots\ldots$$

$$(5)\qquad \frac{d}{dt}\frac{\partial T}{\partial P} - R\frac{\partial T}{\partial Q} + Q\frac{\partial T}{\partial R} - W\frac{\partial T}{\partial V} + V\frac{\partial T}{\partial W} = 0, \ \ldots\ldots, \ \ldots\ldots$$

having three first integrals, and leading to the same result as before, but not so simply and directly.

4. Darboux's notation employed in the corresponding motion of the top (Despeyrous—*Mécanique*) is convenient here also; using L' instead of his B, we put

$$(1)\qquad \frac{n'r}{Fn} = 2\frac{L}{M}, \quad \frac{Fry_3}{Fn} = 2\frac{L'}{M}, \quad \frac{h}{F} = -4\frac{B}{M},$$

and now

$$(2)\qquad \begin{cases} Z = a(z^2-1)\left(z^2 - 4\dfrac{B}{M}z - \dfrac{m_1}{m}\right) - 4\left(\dfrac{L'z-L}{M}\right)^2 \\ \quad = a(z^2-1)\left(z^2 - 4\dfrac{B}{M}z - 1 - aD\right) - 4\left(\dfrac{L'z-L}{M}\right)^2, \end{cases}$$

or

$$(3)\qquad \begin{cases} Z = a(z^2-1)\left(z^2 - 4\dfrac{B}{M}z - \dfrac{m_2}{m}\right) - 4\left(\dfrac{Lz-L'}{M}\right)^2 \\ \quad = a(z^2-1)\left(z^2 - 4\dfrac{B}{M}z - 1 - aE\right) - 4\left(\dfrac{Lz-L'}{M}\right)^2, \end{cases}$$

where

$$(4)\qquad 1 + aD = \frac{m_1}{m}, \quad 1 + aE = \frac{m_2}{m},$$

$$(5)\qquad a\frac{m_1 - m_2}{m} = D - E = 4\frac{L^2 - L'^2}{M^2}.$$

Putting

$$(6)\qquad Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e,$$

then by analogy with the corresponding case of the spinning top

$$(7)\qquad \begin{cases} Z = a(z^2-1)(z^2 + 4abz - 1 - aD) - 4\left(\dfrac{L'z-L}{M}\right)^2 \\ \quad = a(z^2-1)(z^2 + 4abz - 1 - aE) - 4\left(\dfrac{Lz-L'}{M}\right)^2, \end{cases}$$

where

(8) $$D + 4\frac{L'^2}{M^2} = E + 4\frac{L^2}{M^2} = F' \text{ suppose}$$

and

(9) $$\begin{cases} D = a\left(\frac{m_1}{m} - 1\right),\ E = a\left(\frac{m_2}{m} - 1\right), \\ ab = -\frac{B}{M},\ b = -a\frac{B}{M}. \end{cases}$$

The special case of h, b, $B = 0$ was the one considered by Kirchhoff, and in the *American Journal of Mathematics.*

But now we continue with the more general case invented by Clebsch and discussed by Halphen, and identify the notation as we go on.

Proceeding with the determination of the Eulerian angles, ϑ, ϕ, ψ, by means of the Table above,

(10) $$\cos AX + i\cos AY = (\cos\phi + i\cos\vartheta\sin\phi)e^{\psi i} = \alpha^2 - \beta^2,$$

(11) $$\cos AZ = \sin\vartheta\sin\phi = \beta\delta - \alpha\gamma,$$

(12) $$\cos BX + i\cos BY = (-\sin\phi + i\cos\vartheta\cos\phi)e^{\psi i} = i(\alpha^2 + \beta^2),$$

(13) $$\cos BZ = \sin\vartheta\cos\phi = i(-\beta\delta - \alpha\gamma),$$

(14) $$\cos CX + i\cos CY = -i\sin\vartheta e^{\psi i} = -2\alpha\beta,$$

(15) $$\cos CZ = \cos\vartheta = \alpha\delta + \beta\gamma.$$

So also

(16) $$\cos AX + i\cos BX = (\cos\psi - i\cos\vartheta\sin\psi)e^{-\phi i} = -\beta^2 + \delta^2,$$

(17) $$\cos CX = \sin\vartheta\sin\psi = -\alpha\beta + \gamma\delta,$$

(18) $$\cos AY + i\cos BY = (\sin\psi + i\cos\vartheta\cos\psi)e^{-\phi i} = i(\beta^2 + \delta^2),$$

(19) $$\cos CY = -\sin\vartheta\cos\psi = i(\alpha\beta + \gamma\delta),$$

(20) $$\cos AZ + i\cos BZ = i\sin\vartheta e^{-\phi i} = 2\beta\delta,$$

(21) $$\cos CZ = \cos\vartheta = \alpha\delta + \beta\gamma.$$

Again

(22) $$R = 2i\left(\alpha\frac{d\delta}{dt} - \frac{d\beta}{dt}\gamma\right),$$

(23) $$P + Qi = \left(\frac{d\vartheta}{dt} + i\sin\vartheta\frac{d\psi}{dt}\right)e^{-\phi i} = 2i\left(\beta\frac{d\delta}{dt} - \frac{d\beta}{dt}\delta\right),$$

$$(24)\qquad \left\{\begin{aligned} r(y_1+y_2 i) &= P + Qi - q(x_1+x_2 i)\\ &= \left[\frac{d\vartheta}{dt} + i\sin\vartheta\left(\frac{d\psi}{dt} - qF\right)\right]e^{-\phi i}\\ &= 2i\left(\beta\frac{d\delta}{dt} - \frac{d\beta}{dt}\delta\right) - 2qF\beta\delta, \end{aligned}\right.$$

$$(25)\qquad ry_1 = \cos\phi\frac{d\vartheta}{dt} + \sin\vartheta\sin\phi\left(\frac{d\psi}{dt} - qF\right),$$

$$(26)\qquad ry_2 = -\sin\phi\frac{d\vartheta}{dt} + \sin\vartheta\cos\phi\left(\frac{d\psi}{dt} - qF\right),$$

$$(27)\qquad \left\{\begin{aligned} n' - x_3 y_3 &= x_1 y_1 + x_2 y_2\\ &= F\sin\vartheta(y_1\sin\phi + y_2\cos\phi)\\ &= \frac{F}{r}\sin^2\vartheta\left(\frac{d\psi}{dt} - qF\right), \end{aligned}\right.$$

so that, with $x_3 = Fz$,

$$(28)\qquad \frac{d\psi}{dt} - qF = \frac{n'r - Fry_3 z}{F(1-z^2)} = 2n\frac{L - L'z}{M(1-z^2)},$$

$$(29)\qquad \psi - qFt = 2\int\frac{L - L'z}{M(1-z^2)}\frac{dz}{\sqrt{Z}},$$

introducing the III elliptic integral.

From (15), §1,

$$(30)\qquad \left\{\begin{aligned} \frac{d\phi}{dt} &= R - \cos\vartheta\frac{d\psi}{dt}\\ &= q'x_3 + r'y_3 - z\left(qF + \frac{2n}{M}\frac{L - L'z}{1-z^2}\right)\\ &= (q'-q)Fz + (r'-r)y_3 + 2n\frac{L'}{M} - 2\frac{n}{M}\frac{Lz - L'z^2}{1-z^2}\\ &= (q'-q)Fz + (r'-r)y_3 + 2n\frac{L' - Lz}{M(1-z^2)} \end{aligned}\right.$$

$$(31)\qquad \phi = (q'-q)\frac{F}{n}\int\left(z + \frac{r'-r}{q'-q}\frac{y_3}{F}\right)\frac{dz}{\sqrt{Z}} + 2\int\frac{L' - Lz}{M(1-z^2)}\frac{dz}{\sqrt{Z}},$$

introducing another III elliptic integral, with parameter (w) corresponding to $z = \infty$.

Thence

$$(32)\quad \phi + \psi = \frac{F}{n}\int\left[(q'-q)z + (r'-r)\frac{y_3}{F} + q\right]\frac{dz}{\sqrt{Z}} + 2\int\frac{L + L'}{M(1+z)}\frac{dz}{\sqrt{Z}},$$

$$(33)\quad \phi - \psi = \frac{F}{n}\int\left[(q'-q)z + (r'-r)\frac{y_3}{F} - q\right]\frac{dz}{\sqrt{Z}} - 2\int\frac{L - L'}{M(1-z)}\frac{dz}{\sqrt{Z}}.$$

From (24) and (27), §1,

$$(34)\qquad \frac{y_1 + y_2 i}{x_1 + x_2 i} = \frac{\dfrac{d\vartheta}{dt} + i \sin\vartheta\left(\dfrac{d\psi}{dt} - qF\right)}{iFr \sin\vartheta}$$

and thence Halphen's Φ (F. E. II, p. 158), employing his notation for ρ, s for a moment, is given by

$$(35)\qquad \left\{\begin{aligned} \frac{\rho^2}{s} y_3\Phi &= (x_1^2 + x_2^2)\frac{y_1 + y_2 i}{x_1 + x_2 i} \\ &= F^2 \sin^2\vartheta \frac{-i\dfrac{d\vartheta}{dt} + \sin\vartheta\left(\dfrac{d\psi}{dt} - qF\right)}{Fr \sin\vartheta} \\ &= \frac{F}{r}\left(i\frac{dz}{dt} + 2n\frac{L - L'z}{M}\right) \\ &= \frac{Fn}{r}\left(i\sqrt{Z} + 2\frac{L - L'z}{M}\right). \end{aligned}\right.$$

5. For the motion of translation, denote by X, Y, Z, the coordinates of the origin O fixed in the body, with respect to axes fixed in space and parallel to OX, OY, OZ; then according to Kirchhoff's equations (Halphen, F. E. II, p. 162),

$$(1)\qquad FX = y_1 \cos AY + y_2 \cos BY + y_3 \cos CY,$$

$$(2)\qquad FY = -y_1 \cos AX - y_2 \cos BX - y_3 \cos CX;$$

so that, from the preceding relations

$$(3)\qquad \begin{aligned} F(X + Yi) = -i\,[y_1(\cos\phi + i\cos\vartheta\sin\phi) &+ \\ y_2(-\sin\phi + i\cos\vartheta\cos\phi) &- iy_3\sin\vartheta]\,e^{\psi i} \end{aligned}$$

$$(4)\qquad \left\{\begin{aligned} Fr(X + Yi) &= \left[-i\frac{d\vartheta}{dt} + \sin\vartheta\cos\vartheta\left(\frac{d\psi}{dt} - qF\right) - 2n\frac{L'}{M}\sin\vartheta\right]e^{\psi i} \\ &= \left(-i\frac{d\vartheta}{dt} + 2n\frac{Lz - L'}{M\sin\vartheta}\right)e^{\psi i}. \end{aligned}\right.$$

Changing to polar coordinates ρ, ϖ in a plane perpendicular to OZ, such that

$$(5)\qquad \frac{Fr}{n}X = \rho\cos\varpi,\ \frac{Fr}{n}Y = \rho\sin\varpi,$$

$$(6)\qquad \rho\exp(\varpi - \psi)i = \frac{1}{\sin\vartheta}\left(i\sqrt{Z} + 2\frac{Lz - L'}{M}\right),$$

so that

$$(7)\qquad \rho^2 = -a\left(z^2 - 4\frac{B}{M}z - 1 - aE\right),$$

(8) $\rho \sin \vartheta \sin(\varpi - \psi) = \surd Z,$

(9) $\rho \sin \vartheta \cos(\varpi - \psi) = 2\dfrac{Lz - L'}{M},$

(10) $$\cot(\varpi - \psi) = 2\frac{Lz - L'}{M\surd Z},$$

(11) $$\varpi = \psi + \cot^{-1} 2\frac{Lz - L'}{M\surd Z},$$

so that ϖ and ψ depend on the same III elliptic integral; the parameter of which will be denoted by v.

Differentiating

(12) $$\left\{\begin{aligned} \frac{d\varpi}{dt} &= qF + 2n\frac{-L'z + L}{M(-z^2 + 1)} + n\surd Z\frac{d}{dz}\cot^{-1} 2\frac{Lz - L'}{M\surd Z} \\ &= qF + 2n\frac{\left(z - 2\dfrac{B}{M}\right)\dfrac{Lz - L'}{M}}{z^3 - 4\dfrac{B}{M} - 1 - aE}, \end{aligned}\right.$$

(13) $$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{\left(z - 2\dfrac{B}{M}\right)n\surd Z}{z^3 - 4\dfrac{B}{M}z - 1 - aE},$$

(14) $$\rho\frac{d(\varpi - qFt)}{d\rho} = 2\frac{Lz - L'}{M\surd Z} = \cot(\varpi - \psi),$$

so that the plane CZ is parallel to the tangent to the curve $(\rho,\ \varpi - qFt)$; and the velocity along this curve

(15) $$\rho\sqrt{\left[\left(\frac{1}{\rho}\frac{d\rho}{dt}\right)^2 + \left(\frac{d\varpi}{dt} - qF\right)^2\right]} = n\left(z - 2\frac{B}{M}\right)\sin\vartheta.$$

Thus the hodograph of the curve $(\rho,\ \varpi - qFt)$ is given by

(16) $$\frac{d}{dt}(\rho e^{\varpi i - qFti}) = n\left(z - 2\frac{B}{M}\right)\sin\vartheta e^{\psi i - qFti} = \tfrac{1}{2}an\frac{d\rho^2}{d\vartheta}e^{\psi i - qFti}.$$

The other quantities such as $x_1 + x_2 i$, $y_1 + y_2 i$, $U + Vi$, $P + Qi$ depend on $e^{\phi i}$, and ϕ is given in (31), §4, by two elliptic integrals of the III kind, one with parameter corresponding to $z = \infty$ and denoted by w, the other depending on a parameter denoted by v'.

If v_1 and v_2 are the parameters corresponding to $z = +1$ and -1, equation (32) and (33), §4, show that

(17) $$v = v_1 + v_2, \quad v' = v_1 - v_2,$$

(18) $$v_1 = \tfrac{1}{2}(v + v'), \quad v_2 = \tfrac{1}{2}(v - v').$$

To prove Kirchhoff's expressions for X and Y in (1) and (2), we return to the use of U, V, W, P, Q, R as independent variables, and to the expression of the kinetic energy as a quadratic function of these variables as in §3.

Denoting

$$\cos AX,\ \cos BX,\ \cos CX \text{ by } \alpha_1,\ \alpha_2,\ \alpha_3; \tag{19}$$

$$\cos AY,\ \cos BY,\ \cos CY \text{ by } \beta_1,\ \beta_2,\ \beta_3; \tag{20}$$

$$\cos AZ,\ \cos BZ,\ \cos CZ \text{ by } \gamma_1,\ \gamma_2,\ \gamma_3; \tag{21}$$

then

$$\left\{\begin{aligned} \frac{dX}{dt} &= \alpha_1 U + \alpha_2 V + \alpha_3 W \\ &= (\beta_2\gamma_3 - \beta_3\gamma_2)U + (\beta_3\gamma_1 - \beta_1\gamma_3)V + (\beta_1\gamma_2 - \beta_2\gamma_1)W \\ &= \beta_1(W\gamma_2 - V\gamma_3) + \beta_2(U\gamma_3 - W\gamma_1) + \beta_3(V\gamma_1 - U\gamma_2) \end{aligned}\right. \tag{22}$$

and then, since

$$\frac{\partial T}{\partial U} = x_1 = F\gamma_1,\ \frac{\partial T}{\partial V} = x_2 = F\gamma_2,\ \frac{\partial T}{\partial W} = x_3 = F\gamma_3, \tag{23}$$

and (Halphen, F. E. II, p. 148)

$$\frac{d}{dt}\frac{\partial T}{\partial P} = W\frac{\partial T}{\partial V} - V\frac{\partial T}{\partial W} + R\frac{\partial T}{\partial Q} - Q\frac{\partial T}{\partial R} \tag{24}$$

we find

$$\left\{\begin{aligned} F\frac{dX}{dt} &= \beta_1\left(W\frac{\partial T}{\partial V} - V\frac{\partial T}{\partial W}\right) + \beta_2\left(U\frac{\partial T}{\partial W} - W\frac{\partial T}{\partial U}\right) \\ &\qquad + \beta_3\left(V\frac{\partial T}{\partial U} - U\frac{\partial T}{\partial V}\right) \\ &= \beta_1\frac{d}{dt}\frac{\partial T}{\partial P} - \beta_1\left(R\frac{\partial T}{\partial Q} - Q\frac{\partial T}{\partial R}\right) \\ &\quad + \beta_2\frac{d}{dt}\frac{\partial T}{\partial Q} - \beta_2\left(P\frac{\partial T}{\partial R} - R\frac{\partial T}{\partial P}\right) \\ &\quad + \beta_3\frac{d}{dt}\frac{\partial T}{\partial R} - \beta_3\left(Q\frac{\partial T}{\partial P} - P\frac{\partial T}{\partial Q}\right) \\ &= \beta_1\frac{d}{dt}\frac{\partial T}{\partial P} + \beta_2\frac{d}{dt}\frac{\partial T}{\partial Q} + \beta_3\frac{d}{dt}\frac{\partial T}{\partial R} \\ &\quad + (\beta_2 R - \beta_3 Q)\frac{\partial T}{\partial P} + (\beta_3 P - \beta_1 R)\frac{\partial T}{\partial Q} + (\beta_1 Q - \beta_2 P)\frac{\partial T}{\partial R} \\ &= \beta_1\frac{d}{dt}\frac{\partial T}{\partial P} + \beta_2\frac{d}{dt}\frac{\partial T}{\partial Q} + \beta_3\frac{d}{dt}\frac{\partial T}{\partial R} \\ &\quad + \frac{d\beta_1}{dt}\frac{\partial T}{\partial P} + \frac{d\beta_2}{dt}\frac{\partial T}{\partial Q} + \frac{d\beta_3}{dt}\frac{\partial T}{\partial R} \\ &= \frac{d}{dt}\left(\beta_1\frac{\partial T}{\partial P} + \beta_2\frac{\partial T}{\partial Q} + \beta_3\frac{\partial T}{\partial R}\right); \end{aligned}\right. \tag{25}$$

and integrating

$$\left\{\begin{aligned} FX &= \beta_1 \frac{\partial T}{\partial P} + \beta_2 \frac{\partial T}{\partial Q} + \beta_3 \frac{\partial T}{\partial R} \\ &= \beta_1 y_1 + \beta_2 y_2 + \beta_3 y_3. \end{aligned}\right. \tag{26}$$

Similarly

$$\left\{\begin{aligned} \frac{dY}{dt} &= \beta_1 U + \beta_2 V + \beta\, W \\ &= (\gamma_2\alpha_3 - \gamma_3\alpha_2)\,U + (\gamma_3\alpha_1 - \gamma_1\alpha_3)\,V + (\gamma_1\alpha_2 - \gamma_2\alpha_1)\,W \\ &= -\alpha_1(W\gamma_2 - V\gamma_3) - \alpha_2(U\gamma\alpha_3 - W\gamma_1) - {}_3(V\gamma_1 - U\gamma_2) \end{aligned}\right. \tag{27}$$

$$\left\{\begin{aligned} F\frac{dY}{dt} &= -\alpha_1\left(W\frac{\partial T}{\partial V} - V\frac{\partial T}{\partial W}\right) - \alpha_2\left(U\frac{\partial T}{\partial W} - W\frac{\partial T}{\partial U}\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad - \alpha_3\left(V\frac{\partial T}{\partial U} - U\frac{\partial T}{\partial V}\right) \\ &= -\alpha_1\frac{d}{dt}\frac{\partial T}{\partial P} + \alpha_1\left(R\frac{\partial T}{\partial Q} - Q\frac{\partial T}{\partial R}\right) \\ &\quad - \alpha_2\frac{d}{dt}\frac{\partial T}{\partial Q} + \alpha_2\left(P\frac{\partial T}{\partial R} - R\frac{\partial T}{\partial P}\right) \\ &\quad - \alpha_3\frac{d}{dt}\frac{\partial T}{\partial R} + \alpha_3\left(Q\frac{\partial T}{\partial P} - P\frac{\partial T}{\partial Q}\right) \\ &= -\alpha_1\frac{d}{dt}\frac{\partial T}{\partial P} - \alpha_2\frac{d}{dt}\frac{\partial T}{\partial Q} - \alpha_3\frac{d}{dt}\frac{\partial T}{\partial R} \\ &\quad - (\alpha_2 R - \alpha_3 Q)\frac{\partial T}{\partial P} - (\alpha_3 P - \alpha_1 R)\frac{\partial T}{\partial Q} - (\alpha_1 Q - \alpha_2 P)\frac{\partial T}{\partial R} \\ &= -\alpha_1\frac{d}{dt}\frac{\partial T}{\partial P} - \alpha_2\frac{d}{dt}\frac{\partial T}{\partial Q} - \alpha_3\frac{d}{dt}\frac{\partial T}{\partial R} \\ &\quad - \frac{d\alpha_1}{dt}\frac{\partial T}{\partial P} - \frac{d\alpha_2}{dt}\frac{\partial T}{\partial Q} - \frac{d\alpha_3}{dt}\frac{\partial T}{\partial R} \\ &= -\frac{d}{dt}\left(\alpha_1\frac{\partial T}{\partial P} + \alpha_2\frac{\partial T}{\partial Q} + \alpha_3\frac{\partial T}{\partial R}\right); \end{aligned}\right. \tag{28}$$

and integrating

$$\left\{\begin{aligned} FY &= -\alpha_1\frac{\partial T}{\partial P} - \alpha_2\frac{\partial T}{\partial Q} - \alpha_3\frac{\partial T}{\partial R} \\ &= -\alpha_1 y_1 - \alpha_2 y_2 - \alpha_3 y_3 \end{aligned}\right. \tag{29}$$

(Riemann-Weber, *Partielle Differentialgleichungen*).

As for the coordinate Z (which should be distinguished by an accent from the previous use of Z),

$$(30)\quad \left\{\begin{aligned} \frac{dZ'}{dt} &= U\cos AZ + V\cos BZ + W\cos CZ \\ &= (px_1 + qy_1)\sin\vartheta\sin\phi + (px_2 + qy_2)\sin\vartheta\cos\phi \\ &\qquad\qquad + (p'x_3 + q'y_3)\cos\vartheta) \\ &= pF\sin^2\vartheta + p'F\cos^2\vartheta + q\sin\vartheta(y_1\sin\phi + y_2\cos\phi) + q'y_3\cos\vartheta \\ &= (p'-p)Fz^2 + pF + \frac{q}{r}\sin^2\vartheta\left(\frac{d\psi}{dt} - qF\right) + q'y_3z \\ &= (p'-p)Fz^2 + pF + q\left(\frac{n'}{F} - y_3z\right) + q'y_3z \\ &= (p'-p)Fz^2 + (q'-q)y_3z + pF + \frac{qn'}{F} \\ &= (p'-p)F\left(z - \frac{B}{M}\right)^2 - \tfrac{1}{16}(p'-p)\frac{p^2}{F} + pF + \frac{qn'}{F} \end{aligned}\right.$$

so that Z' depends on an elliptic integral of the second kind, to be given hereafter, in §8.

6. In the special case of $p - p' = 0$,

$$(1)\qquad \frac{dz^2}{dt^2} = 2n^2Z,\ n^2 = (q'-q)Fry_3,$$

$$(2)\qquad \left\{\begin{aligned} Z &= (z^2-1)(z-D) - 2\left(\frac{L'z - L}{M}\right)^2 \\ &= (z^2-1)(z-E) - 2\left(\frac{Lz - L'}{M}\right)^2 \end{aligned}\right.$$

and the motion of the axis can be compared directly with that of a symmetrical top moving about its point under gravity, while the curve $(\rho,\ \varpi - qFt)$ is traced out by the vector of its angular momentum.

Also

$$(3)\qquad \frac{dZ'}{dt} = (q'-q)y_3z + pF + \frac{qn'}{F} = \frac{n^2z}{Fr} + pF + \frac{qn'}{F}.$$

When $q - q' = 0$, too, the motion is non-elliptic; and when $r - r' = 0$ as well, we have the case "helicoidally isotropic," and the centre describes a uniform helix.

When $m = 0$, $F = 0$, the impulse of the motion reduces to a couple; and

$x_1, x_2, x_3 = 0$,

(4) $$\begin{cases} U = qy_1, & V = qy_2, & W = q'y_3, \\ P = ry_1, & Q = ry_2, & R = r'y_3; \end{cases}$$

and the three equations (3), §1, expressing the constancy in direction of the vector resultant of y_1, y_2, y_3, reduce to

(5) $$\frac{dy_1}{dt} = (r' - r)y_2y_3, \quad \frac{dy_2}{dt} = -(r' - r)y_1y_3, \quad \frac{dy_3}{dt} = 0.$$

Taking OZ in the direction of the resultant couple G,

(6) $y_3 = G \cos \vartheta$, so that ϑ is constant,

(7) $$y_1 = G \sin \vartheta \sin \phi = y_3 \tan \vartheta \sin \phi, \quad y_2 = G \sin \vartheta \cos \phi = y_3 \tan \vartheta \cos \phi,$$

(8) $$\frac{d\phi}{dt} = (r' - r)y_3, \quad \cos \vartheta \frac{d\psi}{dt} = ry_3,$$

(9) $$\frac{dX}{dt} = -(q - q')y_3 \sin \vartheta \sin \psi, \quad \frac{dY}{dt} = (q - q')y_3 \sin \vartheta \cos \psi,$$

(10) $$X = \frac{q - q'}{r} \sin \vartheta \cos \vartheta \cos \psi, \quad Y = \frac{q - q'}{r} \sin \vartheta \cos \vartheta \sin \psi,$$

(11) $$\frac{dZ'}{dt} = (q \sin^2 \vartheta + q' \cos^2 \vartheta) G, \quad Z' = (q \sin^2 \vartheta - q' \cos^2 \vartheta) Gt;$$

and the motion of O is helicoidal.

There is an elliptic function solution in the most general case of a body of any shape in the liquid when the resultant impulse of the motion reduces to a couple; the equations of motion reduce to Euler's form for no applied forces, as if the liquid was absent, and a Poinsot geometrical interpretation can be constructed (Lamb, *Proc. London Math. Society*, Vol. VIII; Love, *Proc. Cambridge Phil. Society*, 1889).

7. We now proceed to the Elliptic Function solution, denoting the elliptic argument by u, where

(1) $$u = \int \frac{dz}{\sqrt{Z}},$$

and Z has the form in equation (2), (3), (6), §4.

Employ the Biermann-Weierstrass formula

(2) $$\wp(u_1 \pm u_2) = \frac{F(z_1, z_2) \mp \sqrt{Z_1}\sqrt{Z_2}}{2(z_1 - z_2)^2}$$

where

(3) $$F(z_1, z_2) = az_1^2z_2^2 + 2bz_1z_2(z_1 + z_2) + c(z_1^2 + 4z_1z_2 + z_2^2) + 2d(z_1 + z_2) + e.$$

Then if v_1 and v_2 correspond to $z_1 = +1$ and $z_2 = -1$,

(4) $$\wp(v_1 \pm v_2) = \frac{1}{8}\left(a - 2c + e \mp 4\frac{L^2 - L'^2}{M^2}\right);$$

so that, putting

(5) $$v_1 + v_2 = v, \quad v_1 - v_2 = v',$$

(6) $$6\wp v = 2a + E - 2\frac{L^2}{M^2},$$

(7) $$6\wp v' = 2a + D - 2\frac{L'^2}{M^2},$$

(8) $$\wp v' - \wp v = \frac{L^2 - L'^2}{M^2} = \tfrac{1}{4}(D - E).$$

Another Biermann-Weierstrass formula gives

(9) $$\left\{\begin{aligned} i\wp'(v_1 \pm v_2) = {} & \frac{L - L'}{M}(-a + 2b - 2d + e) \\ & \mp \frac{L + L'}{M}(-a - 2b + 2d + e), \end{aligned}\right.$$

so that

(10) $$i\wp' v = \frac{EL}{2M} + 2a\frac{BL'}{M^2},$$

(11) $$i\wp' v' = \frac{DL'}{2M} + 2a\frac{BL}{M^2},$$

Again if v_3, v_4 correspond to

(12) $$\rho = 0, \quad z = 2\frac{B}{M} \pm \sqrt{1 + aE + 4\frac{B^2}{M^2}},$$

we shall find

(13) $$6\wp(v_3 + v_4) = 2a + E - 2\frac{L^2}{M^2},$$

so that

(14) $$v_3 + v_4 = v = v_1 + v_2,$$

as could be anticipated from equation (3), §4.

When a root e of the discriminating cubic

(15) $$4s^3 - g_2 s - g_3 = 0$$

is known, and thence the resolution of Z into quadratic factors, say $Z = XY$, then $\wp(u_1 \pm u_2) - e$ is a square, which can be written

(16) $$\wp(u_1 \pm u_2) - e = \tfrac{1}{4}a\left(\frac{\sqrt{X_1}\sqrt{Y_2} \mp \sqrt{X_2}\sqrt{Y_1}}{z_1 - z_2}\right)^2,$$

a formula due to Laguerre (*Bulletin de la Société mathématique*, 1875).

If $u = w$ makes $z = \infty$, then from Hermite's formula, H denoting the Hessian of Z,

$$(17) \qquad \wp 2u = \frac{H}{-Z}, \quad H = (ac - b^2)z^4 + \ldots,$$

$$(18) \qquad \wp 2w = \frac{ac - b^2}{-a} = \tfrac{1}{3}a + \tfrac{1}{6}E + \tfrac{2}{3}\frac{L^2}{M^2} + a\frac{B^2}{M^2},$$

$$(19) \qquad a^2\wp' 2w = a^2 d - 3abc + 2b^3,$$

$$(20) \qquad \wp 2w - \wp v = \frac{aB^2 + L^2}{M^2},$$

$$(21) \qquad \wp 2w - \wp v' = \frac{aB^2 + L'^2}{M^2}.$$

Supposing Z is resolved into factors

$$(22) \qquad Z = a(z - \alpha)(z - \beta)(z - \gamma)(z - \delta)$$

and that $u = 0$ corresponds to $z = \delta$, the Biermann-Weierstrass formula (2) gives

$$(23) \qquad \begin{cases} \wp u = \dfrac{a\delta^2 z^2 + 2b\delta z(z + \delta) + c(z^2 + 4\delta z + \delta^2) + 2d(z + \delta) + e}{2(z - \delta)^2} \\[2ex] \quad = \dfrac{a\delta^3 + 3b\delta^2 + 3c\delta + d}{z - \delta} + \tfrac{1}{2}(a\delta^2 + 2b\delta + c) \\[2ex] \quad = \dfrac{\frac{1}{4}Z'(\delta)}{z - \delta} + \tfrac{1}{24}Z''(\delta), \end{cases}$$

a fundamental formula; and

$$(24) \qquad \wp' u = -\frac{\frac{1}{4}Z'(\delta)}{(z - \delta)^2}\frac{dz}{du} = -\tfrac{1}{4}Z'(\delta)\frac{\sqrt{Z}}{(z - \delta)^2}.$$

Putting $u = w$, $z = \infty$,

$$(25) \qquad \wp w = \tfrac{1}{24}Z''(\delta) = \tfrac{1}{2}(a\delta^2 + 2b\delta + c),$$

$$(26) \qquad \wp u - \wp w = \frac{\frac{1}{4}Z'(\delta)}{z - \delta} = \frac{a\delta^3 + 3b\delta^2 + 3c\delta + d}{z - \delta},$$

$$(27) \qquad \wp' w = -\tfrac{1}{4}Z'(\delta)\sqrt{a},$$

$$(28) \qquad \wp' u = \frac{\wp' w}{\sqrt{a}}\frac{\sqrt{Z}}{(z - \delta)^2}$$

and differentiating again

$$(29) \quad \wp'' u = \frac{2\wp' w}{\sqrt{a}}\frac{(a\delta + b)z^2 + (a\delta^2 + 4b\delta + 3c)z + a\delta^3 + 4b\delta^2 + 6c\delta + 3d}{(z - \delta)^2},$$

$$(30) \quad \wp'' w = \frac{2\wp' w}{\sqrt{a}}(a\delta + b).$$

Then

$$(31)\qquad \sqrt{a}(z-\delta)=\frac{-\wp' w}{\wp u-\wp w},$$

and by Laguerre's formula (16)

$$(32)\qquad \wp u-e_\alpha=\tfrac{1}{4}a(\delta-\beta)(\delta-\gamma)\frac{(z-\alpha)}{(z-\delta)},$$

$$(33)\qquad \wp w-e_\alpha=\tfrac{1}{4}a(\delta-\beta)(\delta-\gamma),$$

$$(34)\qquad \frac{\wp u-e_\alpha}{\wp w-e_\alpha}=\frac{z-\alpha}{z-\delta},$$

so that

$$(35)\qquad \sqrt{a}(z-\alpha)=\frac{-\wp' w}{\wp u-\wp w}\,\frac{\wp u-e_\alpha}{\wp w-e_\alpha};$$

and by differentiation of (31)

$$(36)\qquad \begin{cases} \sqrt{a}\,\dfrac{dz}{du}=\sqrt{a}\sqrt{Z}=\dfrac{\wp' u\wp' w}{(\wp u-\wp w)^2} \\ \qquad =\wp(u-w)-\wp(u+w). \end{cases}$$

Proceeding with the differentiation

$$(37)\qquad \begin{cases} az^3+3bz^2+3cz+d=\tfrac{1}{4}\dfrac{dZ}{dz}=\tfrac{1}{2}\dfrac{d\sqrt{Z}}{du} \\ \qquad =\dfrac{\wp'(u-w)-\wp'(u+w)}{2\sqrt{a}}, \end{cases}$$

$$(38)\qquad \begin{cases} az^2+2bz+c=\tfrac{1}{12}\dfrac{d^2Z}{dz^2}=\dfrac{1}{6\sqrt{Z}}\,\dfrac{d^2\sqrt{Z}}{du^2} \\ \qquad =\tfrac{1}{6}\dfrac{\wp''(u-w)-\wp''(u+w)}{\wp(u-w)-\wp(u+w)} \\ \qquad =\wp(u-w)+\wp(u+w), \end{cases}$$

$$(39)\qquad \begin{cases} (az+b)^2=a[\wp(u-w)+\wp(u+w)+\wp 2w] \\ \qquad =\tfrac{1}{4}a\left[\dfrac{\wp'(u-w)+\wp'(u+w)}{\wp(u-w)-\wp(u+w)}\right]^2, \end{cases}$$

$$(40)\qquad \begin{cases} az+b=\tfrac{1}{2}[\wp'(u-w)+\wp'(u+w)]\dfrac{du}{dz} \\ \qquad =\tfrac{1}{2}\sqrt{a}\,\dfrac{\wp'(u-w)+\wp'(u+w)}{\wp(u-w)-\wp(u+w)} \\ \qquad =\sqrt{a}[\zeta(u+w)-\zeta(u-w)-\zeta 2w], \end{cases}$$

and with $z = \delta$, $u = 0$,

$$(41) \qquad a\delta + b = \sqrt{a}\,(2\zeta w - \zeta 2w) = \tfrac{1}{2}\sqrt{a}\,\frac{\wp'' w}{\wp' w}.$$

From (25) and (33), by subtraction,

$$(42) \qquad e_\alpha = \tfrac{1}{12}a\left[(\alpha + \delta)(\beta + \gamma) - 2\alpha\delta - 2\beta\gamma\right]$$

and thence

$$(43) \qquad e_\beta - e_\gamma = \tfrac{1}{4}a(\alpha - \delta)(\beta - \gamma).$$

With $z_1 = z$, $u_1 = u$ and $z_2 = \infty$, $u_2 = w$ in the Biermann-Weierstrass formula (2)

$$(44) \qquad \wp(u + w) = \tfrac{1}{2}(az^2 + 2bz + c - \sqrt{a}\sqrt{Z}),$$

$$(45) \qquad \wp(\omega_\alpha + w) = \tfrac{1}{2}(a\alpha^2 + 2b\alpha + c),$$

supposing $z = \alpha$ makes $u = \omega_\alpha$, a half-period; and then

$$(46) \qquad \wp(\omega_\alpha + w) - e_\alpha = \tfrac{1}{4}a(\alpha - \beta)(\alpha - \gamma);$$

and by analogy with (41),

$$(47) \qquad a\alpha + b = \tfrac{1}{2}\sqrt{a}\,\frac{\wp''(\omega_\alpha + w)}{\wp'(\omega_\alpha + w)}.$$

8. Treating $\dfrac{FrZ'}{n}$ in (30), §5, by analogy with $\dfrac{Fr}{n}(X + Yi)$ or $\rho e^{\varpi i}$ in (5), §5,

$$(1) \qquad \left\{\begin{aligned} \frac{d}{du}\left(\frac{FrZ'}{n}\right) &= \frac{d}{dt}\left(\frac{FrZ'}{n^2}\right) \\ &= az^2 + 2bz + (mp + n'q)\frac{r}{n^2}. \end{aligned}\right.$$

Now with

$$(2) \qquad Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e,$$

and denoting the Hessian of Z by H,

$$(3) \qquad H = \tfrac{1}{12}\frac{d^2Z}{dz^2}Z - \left(\tfrac{1}{4}\frac{dZ}{dz}\right)^2,$$

while

$$(4) \qquad \frac{d^2\sqrt{Z}}{dz^2} = \tfrac{1}{2}\frac{d^2Z}{dz^2}\frac{1}{\sqrt{Z}} - \tfrac{1}{4}\left(\frac{dZ}{dz}\right)^2\frac{1}{Z\sqrt{Z}},$$

so that

$$(5) \qquad -\frac{2H}{Z\sqrt{Z}} + \tfrac{1}{2}\frac{d^2\sqrt{Z}}{dz^2} = \tfrac{1}{12}\frac{d^2Z}{dz^2}\frac{1}{\sqrt{Z}} = \frac{az^2 + 2bz + c}{\sqrt{Z}},$$

and

(6) $$\frac{FrZ'}{n} = -2\int \frac{H}{Z}\frac{dz}{\sqrt{Z}} + \tfrac{1}{2}\frac{d\sqrt{Z}}{dz} - cu + (mp + n'q)\frac{ru}{n^2},$$

in which

(7) $$\frac{H}{Z} = -\wp 2u, \text{ so that } -2\int \frac{H}{Z}\frac{dz}{\sqrt{Z}} = -\zeta 2u,$$

while

(8) $$\begin{cases} \tfrac{1}{2}\dfrac{d\sqrt{Z}}{dz} = \tfrac{1}{2}\dfrac{d\sqrt{Z}}{du} \Big/ \dfrac{dZ}{du} = \tfrac{1}{2}\dfrac{\wp'(u-w) - \wp'(u+w)}{\wp(u-w) - \wp(u+w)} \\ \qquad = \zeta 2u - \zeta(u-w) - \zeta(u+w), \end{cases}$$

(9) $$\frac{FrZ'}{n} = \left(\frac{mpr + n'qr}{n^2} - c\right)u - \zeta(u-w) - \zeta(u+w),$$

the form to employ when $a = +1$, and w is a fraction of the real period. Otherwise with $a = -1$ it is preferable to return to (6) and (7); and then, in the reduction to the Jacobian function of the II stage, where the roots e_1, e_2, e_3 of the discriminating cubic (15), §7, are known, we find

(10) $$\begin{cases} \displaystyle\int (az^2 + 2bz + c)\frac{dz}{\sqrt{Z}} = -2(e_1 - e_3)\int \frac{H + e_2 Z}{H + e_3 Z}\frac{dz}{\sqrt{Z}} \\ \displaystyle\qquad\qquad + \tfrac{1}{4}\sqrt{Z}\frac{d}{dz}\log(H + e_3 Z) + 2e_1\int \frac{dz}{\sqrt{Z}}, \end{cases}$$

as is verified by differentiation.

With the roots in the order $e_1 > e_2 > e_3$,

(11) $$\frac{H + e_2 Z}{H + e_3 Z} = \frac{\wp 2u - e_2}{\wp 2u - e_3} = \text{dn}^2 2Mu, \quad M^2 = e_1 - e_3,$$

(12) $$2(e_1 - e_3)\int \frac{H + e_2 Z}{H + e_3 Z}\frac{dz}{\sqrt{Z}} = 2M^2 \int \text{dn}^2 2Mu\,du = 2M^2\frac{E}{K}u + M\,\text{zn}\,2Mu,$$

so that $\dfrac{FrZ'}{n}$ is expressed by secular terms proportional to u or the time, by an algebraical function $\tfrac{1}{4}\sqrt{Z}\dfrac{d}{dz}\log(H + e_3 Z)$, and by the zeta-function $M\,\text{zn}\,2Mu$, functions which do not become infinite.

9. With $u = v_1,\ v_2$ for $z = +1, -1$,

(1) $$\sqrt{a(1 - z_0)} = \frac{-\wp' w}{\wp v_1 - \wp w},$$

(2) $$\sqrt{a(1 - z)} = \frac{-\wp' w(\wp u - \wp v_1)}{(\wp v_1 - \wp w)(\wp u - \wp w)},$$

and similarly

$$\sqrt{a}(1+z) = \frac{\wp' w(\wp u - \wp v_2)}{(\wp v_2 - \wp w)(\wp u - \wp w)}. \tag{3}$$

Also

$$2i\sqrt{a}\,\frac{L-L'}{M} = \frac{-\wp' v_1 \wp' w}{(\wp v_1 - \wp w)^2} = \wp(v_1 + w) - \wp(v_1 - w), \tag{4}$$

$$2i\sqrt{a}\,\frac{L+L'}{M} = \frac{\wp' v_2 \wp' w}{(\wp v_2 - \wp w)^2} = \wp(v_2 - w) - \wp(v_2 + w), \tag{5}$$

with

$$\psi - qFt = \psi_1 + \psi_2, \tag{6}$$

$$\psi_1 = \frac{L-L'}{M}\int \frac{du}{1-z}, \tag{7}$$

$$\psi_2 = \frac{L+L'}{M}\int \frac{du}{1+z}, \tag{8}$$

$$\left\{\begin{aligned}
\frac{d\psi_1 i}{du} &= \frac{\dfrac{1}{2\sqrt{a}}\,\dfrac{-\wp' v_1 \wp' w}{(\wp v_1 - \wp w)^2}}{\dfrac{1}{\sqrt{a}}\,\dfrac{-\wp' w(\wp u - \wp v_1)}{(\wp v_1 - \wp w)(\wp u - \wp w)}} \\
&= \tfrac{1}{2}\,\frac{\wp' v_1(\wp u - \wp w)}{(\wp v_1 - \wp w)(\wp u - \wp v_1)} \\
&= \frac{\tfrac{1}{2}\wp' v_1}{\wp v_1 - \wp w} + \frac{\tfrac{1}{2}\wp' v_1}{\wp u - \wp v_1} \\
&= \tfrac{1}{2}\zeta(v_1 - w) + \tfrac{1}{2}\zeta(v_1 + w) - \zeta v_1 \\
&\quad + \tfrac{1}{2}\zeta(u - v_1) - \tfrac{1}{2}\zeta(u + v_1) + \zeta v_1.
\end{aligned}\right. \tag{9}$$

Similarly

$$\frac{d\psi_2 i}{du} = \tfrac{1}{2}\zeta(v_2 - w) + \tfrac{1}{2}\zeta(v_2 + w) - \zeta v_2 + \tfrac{1}{2}\zeta(u - v_2) - \tfrac{1}{2}\zeta(u + v_2) + \zeta v_2. \tag{10}$$

Integrating

$$\psi_1 i = \tfrac{1}{2}[\zeta(v_1 - w) + \zeta(v_1 + w)]\,nt + \tfrac{1}{2}\log\frac{\sigma(u - v_1)}{\sigma(u + v_1)}, \tag{11}$$

$$\psi_2 i = \tfrac{1}{2}[\zeta(v_2 - w) + \zeta(v_2 + w)]\,nt + \tfrac{1}{2}\log\frac{\sigma(u - v_2)}{\sigma(u + v_2)}. \tag{12}$$

With $v_1 + v_2 = v$, and the formula

$$\frac{\wp(u - v_1) - \wp(u - v_2)}{\wp(u + v_1) - \wp(u + v_2)} = \frac{\sigma(2u - v)\sigma^2(u + v_1)\sigma^2(u + v_2)}{\sigma(2u + v)\sigma^2(u - v_1)\sigma^2(u - v_2)}, \tag{13}$$

$$(14)\qquad \psi_1 i + \psi_2 i = \tfrac{1}{2} Qnt + \tfrac{1}{2}\log\frac{\sigma(2u-v)}{\sigma(2u+v)} - \tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)},$$

$$(15)\qquad Q = \zeta(v_1-w) + \zeta(v_1+w) + \zeta(v_2-w) + \zeta(v_2+w).$$

Introducing a standard form of the III elliptic integral from the memoir on the subject in the *Phil. Trans.*, 1904,

$$(16)\qquad \begin{cases} I(2u,v) = \displaystyle\int \frac{\frac{P}{M}(s-\sigma) - \frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\,\frac{ds}{\sqrt{S}} \\ \qquad = \tfrac{1}{2} i \left\{\log\dfrac{\sigma(2u+v)}{\sigma(2u-v)} - \left(\dfrac{Pi}{M} + \zeta v\right) 2u\right\}, \end{cases}$$

$$(17)\qquad 2u = \int \frac{ds}{\sqrt{S}},$$

then

$$(18)\begin{cases} (\psi - qFt)i = (\psi_1 + \psi_2)i \\ \qquad = \tfrac{1}{2}\left(Q - 2\zeta v - 2\dfrac{Pi}{M}\right)nt + \tfrac{1}{2} i I(2u,v) - \tfrac{1}{4}\log\dfrac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)}. \end{cases}$$

Another application of the Biermann-Weierstrass formula (2), §7, gives

$$(19)\quad \wp(u \pm v_1) = \frac{(a+2b+c)z^2 + 2(b+2c+d)z + c + 2d + e \pm 2i\dfrac{L-L'}{M}\sqrt{Z}}{2(z-1)^2}$$

$$(20)\quad \wp(u \pm v_2) = \frac{(a-2b+c)z^2 + 2(b-2c+d)z + c - 2d + e \pm 2i\dfrac{L+L'}{M}\sqrt{Z}}{2(z+1)^2}$$

$$(21)\qquad \frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} = \frac{A' + iB'\sqrt{Z}}{A' - iB'\sqrt{Z}},$$

$$(22)\qquad \tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} = \tfrac{1}{2} i \tan^{-1}\frac{B'\sqrt{Z}}{A'},$$

$$(23)\qquad A' = bz^4 + (a+3c)z^3 + 3(b+d)z^2 + (3c+e)z + d,$$

$$(24)\qquad B' = \frac{Lz^2 - 2L'z + L}{M}.$$

A further application of formula (44), §7, gives

$$(25)\qquad \wp(v_1 \pm w) = \frac{a+2b+c}{2} \pm 2i\sqrt{a}\,\frac{L-L'}{M},$$

$$(26)\qquad \wp(v_2 \pm w) = \frac{a-2b+c}{2} \mp 2i\sqrt{a}\,\frac{L+L'}{M},$$

(27) $$\wp(v_1+w)+\wp(v_2-w)=a+c+2i\sqrt{a}\,\frac{L}{M},$$

(28) $$\left\{\begin{aligned}\wp(v_1-w)+\wp(v_2+w)&=a+c-2i\sqrt{a}\,\frac{L}{M}\\ &=-\tfrac{2}{3}\frac{L^2}{M^2}+\tfrac{2}{3}a-\tfrac{1}{6}E-2i\sqrt{a}\,\frac{L}{M},\end{aligned}\right.$$

while

(29) $$\wp(v_1+v_2)=-\frac{L^2}{M^2}+\tfrac{1}{3}a+\tfrac{1}{6}E,$$

and then

(30) $$\begin{aligned}\wp(v_1+w)+\wp(v_2-w)+\wp(v_1+v_2)&=-\frac{L^2}{M^2}+2i\sqrt{a}\,\frac{L}{M}+a\\ &=\left(i\frac{L}{M}+\sqrt{a}\right)^2,\end{aligned}$$

(31) $$\wp(v_1-w)+\wp(v_2+w)+\wp(v_1+v_2)=\left(i\frac{L}{M}-\sqrt{a}\right)^2,$$

(32) $$\zeta(v_1+w)+\zeta(v_2-w)-\zeta(v_1+v_2)=i\frac{L}{M}+\sqrt{a},$$

(33) $$\zeta(v_1-w)+\zeta(v_2+w)-\zeta(v_1+v_1)=i\frac{L}{M}-\sqrt{a},$$

(34) $$\left\{\begin{aligned}Q-2\zeta v&=\zeta(v_1+w)+\zeta(v_2-w)-\zeta(v_1+v_2)\\ &\quad+\zeta(v_1-w)+\zeta(v_2+w)-\zeta(v_1+v_2)\\ &=2i\frac{L}{M},\end{aligned}\right.$$

so that finally,

(35) $$\psi-qFt=\frac{L-P}{M}nt+\tfrac{1}{2}I(2u,v)+\tfrac{1}{2}\tan^{-1}\frac{B'\sqrt{Z}}{A'}.$$

Putting

(36) $$\psi-qFt-\frac{L-P}{M}nt=\psi-pt=\psi',$$

(37) $$\frac{p}{n}=\frac{qF}{n}+\frac{L-P}{M};$$

(38) $$\left\{\begin{aligned}\psi'&=2\int\frac{L-L'z}{M(1-z^2)}\frac{dz}{\sqrt{Z}}-\frac{L-P}{M}\int\frac{dz}{\sqrt{Z}}\\ &=\int\frac{(L-P)z^2-2L'z+L+P}{M(1-z^2)}\frac{dz}{\sqrt{Z}}\\ &=\tfrac{1}{2}I(2u,v)+\tfrac{1}{2}\tan^{-1}\frac{B'\sqrt{Z}}{A'},\end{aligned}\right.$$

and the right-hand side can be made the logarithm of an algebraical function of z when v is chosen as an aliquot μth part of a period, as shown already in the *American Journal of Mathematics*, XX, 1897; and for subsequent details consult the *Phil. Trans.*, 1904, on the Third Elliptic Integral, etc.; to which references in the sequel are indicated by the page and article or section.

For comparison with Halphen's results the following table gives his notation in the left-hand column, and its equivalent on the right as employed in this memoir.

u	$u - w$
$u + v$	$u + w$
v	$2w$
a	$v_1 - w$
b	$-v_2 - w$
$a - b$	v
$a + b + v$	v'
$a + v$	$v_1 + w$
$b + v$	$-v_2 + w$
$u - a$	$u - v_1$
$u + a + v$	$u + v_1$
a_1	$-v_3 - w$
b_1	$v_4 - w$
$-a_1 + b_1$	v
c	$\frac{1}{2}(v_1 - v_3) - w$
c	$-\frac{1}{2}(v_2 - v_4) - w$
a'	$-\frac{1}{2}(v_1 + v_3)$
b'	$\frac{1}{2}(v_2 + v_4)$
w	$\frac{1}{2}(v_1 - v_3) + w$
w	$-\frac{1}{2}(v_2 - v_4) + w$

10. Denoting Klein's functions by α_1, β_1, γ_1, δ_1, in Kirchhoff's special case where q, $q' = 0$, so that

$$\tfrac{1}{2}(\phi + \psi) = \psi_2, \quad \tfrac{1}{2}(\phi - \psi) = -\psi_1, \tag{1}$$

then

$$\alpha_1 = \sqrt{\frac{1 + z}{2}}\, e^{\psi_2 i}, \tag{2}$$

$$(3)\quad \left\{\begin{aligned} \log \alpha_1 &= \psi_2 i + \log\sqrt{\frac{1+z}{2}} \\ &= \tfrac{1}{2}[\zeta(v_2-w)+\zeta(v_2+w)]u \\ &\qquad + \tfrac{1}{2}\log\frac{\sigma(u-v_2)}{\sigma(u+v_2)}\frac{\frac{1}{2}\wp' w(\wp u-\wp v_2)}{\sqrt{a(\wp v_2-\wp w)(\wp u-\wp w)}} \\ &= \tfrac{1}{2}[\zeta(v_2-w)+\zeta(u_2+w)-2\zeta v_2]u \\ &\qquad + \log\frac{\sigma(u-v_2)}{\sigma u\sigma v_2}e^{u\zeta v_2}\sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_2-\wp w)(\wp u-\wp w)}}}, \end{aligned}\right.$$

$$(4)\quad \alpha_1 = e^{l_2 nti}\frac{\sigma(u-v_2)}{\sigma u\sigma v_2}e^{u\zeta v_2}\sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_2-\wp w)(\wp u-\wp w)}}},$$

where

$$(5)\quad l_2 = -\tfrac{1}{2}[\zeta(v_2-w)+\zeta(v_2+w)-2\zeta v_2]i = \frac{-\frac{1}{2}i\wp' v_2}{\wp v_2-\wp w}.$$

Similarly

$$(6)\quad \beta_1 = \sqrt{\frac{-1+z}{2}}e^{\psi_1 i} = e^{l_1 nti}\frac{\sigma(u-v_1)}{\sigma u\sigma v_1}e^{u\zeta v_1}\sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_1-\wp w)(\wp u-\wp w)}}},$$

where

$$(7)\quad l_1 = \frac{-\frac{1}{2}i\wp' v_1}{\wp v_1-\wp w},$$

and

$$(8)\quad \gamma_1 = e^{-l_1 nti}\frac{\sigma(u+v_1)}{\sigma u\sigma v_1}e^{-u\zeta v_1}\sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_1-\wp w)(\wp u-\wp w)}}},$$

$$(9)\quad \delta_1 = e^{-l_2 nti}\frac{\sigma(u+v_2)}{\sigma u\sigma v_2}e^{-u\zeta v_2}\sqrt{\frac{-\frac{1}{2}\wp' w}{\sqrt{a(\wp v_2-\wp w)(\wp u-\wp w)}}}.$$

When μv_1 or μv_2 is congruent to a period, and μ is an odd integer $= 2n+1$ then α_1, β_1, γ_1, or δ_1, is the $(2n+1)$th root of algebraical functions of the form

$$(10)\quad A + iB\sqrt{Z},$$

such that

$$(11)\quad A^2 + B^2 Z = \left(\frac{\pm 1 + z}{2}\right)^{2n+1},$$

and A, B can be determined by the method of *réduites* (Halphen, F. E. II, Chap. 14) or else as explained in *Phil. Trans.*, 1904.

But when μ is even $= 2n$, and nv_1 or nv_2 is congruent to a half-period then α_1, β_1, γ_1, or δ_1, is the nth root of an algebraical function of the form

$$(12)\quad A_1\sqrt{Z_1} + iA_2\sqrt{Z_2},$$

such that

$$(13) \qquad A_1^2 Z_1 + A_2^2 Z_2 = \left(\frac{\pm 1 + z}{2}\right)^n,$$

Z_1 and Z_2 denoting the quadratic factors of Z.

In Clebsch's case where q and q' are not zero, then Klein's functions are given in general by

$$(14) \qquad a = e^{iA} a_1, \quad \beta = e^{-iA'} \beta_1, \quad \gamma = e^{iA'} \gamma_1, \quad \delta = e^{-iA} \delta_1,$$

where from equations (32), (33), §4,

$$(15) \qquad A, A' = \tfrac{1}{2} \frac{F}{n} \int [(q' - q) z + (r' - r) \frac{y_8}{F} \pm q] \, du,$$

and introducing Halphen's notation (F. E. II, p. 157)

$$(16) \qquad \frac{(q' - q) F}{\sqrt{an^2}} = \frac{q' - q}{\sqrt{n(p' - p)}} = -\frac{B}{L} \sqrt{a} = -\beta i,$$

$$(17) \qquad A = -\tfrac{1}{2} \beta i \int \sqrt{a} \left(z - \frac{B}{M}\right) du - \tfrac{1}{2} \beta \sqrt{(-a)} \frac{Bu}{M}$$
$$+ (r' - r) \frac{y_8 u}{2n} + \frac{Fqu}{2n},$$

in which

$$(18) \qquad \left\{ \begin{aligned} \int \sqrt{a} \left(z - \frac{B}{M}\right) du &= \int [\zeta(u + w) + \zeta(u - w) - \zeta 2w] \, du \\ &= \log \frac{\sigma(u + w)}{\sigma(u - w)} e^{-u\zeta 2w}, \end{aligned} \right.$$

so that

$$(19) \qquad e^{iA} = \left[\frac{\sigma(u + w)}{\sigma(u - w)} e^{-u\zeta 2w}\right]^{i\beta} e^{Mui},$$

is the form of the result for A and for A' also; and if $q = q'$, $B = 0$, A and A' reduce to a multiple of the time.

11. As the simplest illustration, suppose v is a half-period, so that

(1) $$\wp' v = 0, \quad \frac{EL}{M} + 4a\frac{BL'}{M^2} = 0,$$

from (10), §7, and then from (3), §4,

(2) $$Z = a(z^2-1)^2 - 4a\frac{B}{L}(z^2-1)\frac{Lz-L'}{M} - 4\left(\frac{Lz-L'}{M}\right)^2;$$

and

(3) $$k^2(z^2-1)^2 - Z = (k^2-a)(z^2-1)^2 + 4a\frac{B}{L}(z^2-1)\frac{Lz-L'}{M} + 4\left(\frac{Lz-L'}{M}\right)^2$$

is made a square if

(4) $$k^2 - a = \frac{B^2}{L^2}, \quad k^2 = \frac{B^2+aL^2}{L^2}.$$

Putting

(5) $$\left\{\begin{aligned} \psi' &= \tfrac{1}{2}\sin^{-1}\frac{\sqrt{Z}}{k(1-z^2)} \\ &= \tfrac{1}{2}\cos^{-1}\frac{a\frac{B}{L}(1-z^2) + 2\frac{Lz-L'}{M}}{k(1-z^2)} \\ &= \begin{matrix}\cos^{-1}\\ \sin^{-1}\end{matrix}\sqrt{\frac{\{\sqrt{(B^2+aL^2)} \pm aB\}(1-z^2) \pm 2L\frac{Lz-L'}{M}}{2\sqrt{(B^2+aL^2)}(1-z^2)}}, \end{aligned}\right.$$

(6) $$\left\{\begin{aligned} \frac{d\psi'}{dz} &= \frac{2\frac{L-L'z}{M} - \frac{L}{M}(1-z^2)}{(1-z^2)\sqrt{Z}} \\ &= \frac{d\psi}{dz} - qF\frac{dt}{dz} - \frac{Ln}{M}\frac{dt}{dz}, \end{aligned}\right.$$

(7) $$\psi - qFt - \frac{L}{M}nt = \psi'.$$

In Kirchhoff's case of $B = 0$, $a = -1$ is to be rejected, and

(8) $$Z_1, Z_2 = z^2 \mp 2\frac{L'z}{M} - 1 \pm 2\frac{L}{M}.$$

Again

(9) $$\rho^2 = -a\left(z^2 - 4\frac{B}{M}z - 1 + 4\frac{B}{M}\frac{L'}{L}\right),$$

(10) $$z^2 - 1 = -4\frac{B}{L}\frac{L'-Lz}{M} - a\rho^2,$$

(11) $$Z = \rho^2\left(a\rho^2 + 4\frac{B}{L}\frac{L' - Lz}{M}\right) - 4\left(\frac{L' - Lz}{M}\right)^2,$$

(12) $$\lambda^2\rho^4 - Z = (\lambda^2 - a)\rho^4 - 4\frac{B}{L}\frac{L' - Lz}{M}\rho^2 + 4\left(\frac{L' - Lz}{M}\right)^2,$$

is a square if

(13) $$\lambda^2 - a = \frac{B^2}{L^2}, \quad \lambda = k;$$

and then putting

(14) $$\varpi' = \tfrac{1}{2}\sin^{-1}\frac{\sqrt{Z}}{k\rho^2} = \tfrac{1}{2}\cos^{-1}\frac{a\frac{B}{L}\rho^2 - 2\frac{L' - Lz}{M}}{k\rho^2},$$

(15) $$\frac{d\varpi'}{dz} = -\frac{2a\left(z - 2\frac{B}{M}\right)\frac{Lz - L'}{M}}{\rho^2\sqrt{Z}} - \frac{L}{M}\frac{1}{\sqrt{Z}} = \frac{d\varpi}{dz} - qF\frac{dt}{dz} - \frac{Ln}{M}\frac{dt}{dz},$$

(16) $$\varpi - qFt - \frac{L}{M}nt = \varpi'.$$

In this case the vector $\sin\vartheta e^{\psi' \iota}$ and $\rho e^{\varpi' \iota}$ each describes the arc of a conic on a plane perpendicular to OZ, and the motion is algebraical except for the secular term $qFt + \frac{L}{M}nt$, and this can be cancelled by putting

(17) $$qFM + Ln = 0.$$

Now suppose in addition that v' is also a half-period so that

(18) $$v_1,\ v_2 = \omega_1 + \tfrac{1}{2}\omega_3,\ \tfrac{1}{2}\omega_3,$$

(19) $$\wp' v' = 0, \quad \frac{DL'}{M} + 4a\frac{BL}{M^2} = 0, \text{ from (11), §7,}$$

(20) $$\frac{D}{E} = \frac{L^2}{L'^2}, \quad D - E = 4\frac{L^2 - L'^2}{M^2},$$

(21) $$D = 4\frac{L^2}{M^2}, \quad E = 4\frac{L'^2}{M^2}, \quad aB = -\frac{LL'}{M}.$$

(22) $$\begin{cases} Z = a(z^2 - 1)\left(z^2 + 4a\frac{LL'}{M^2}z - 1 - 4a\frac{L'^2}{M^2}\right) - 4\left(\frac{Lz - L'}{M}\right)^2 \\ \quad = a\left(z^2 + 2a\frac{LL'}{M} + 1\right)^2 - 4a\left(\frac{L^2}{M^2} + a\right)\left(\frac{L'^2}{M^2} + a\right)z^2. \end{cases}$$

Writing λ, λ' for $\frac{L, L'}{M}$,

(23) $$aZ = (z^2 + 2a\lambda\lambda' z + 1)^2 - 4(\lambda^2 + a)(\lambda'^2 + a)z^2.$$

With $a = 1$, put $\lambda, \lambda' = \operatorname{sh} \alpha, \operatorname{sh} \alpha'$; then

(24) $$Z = Z_1 Z_2,$$

(25) $$Z_1 = z^2 + 2z \operatorname{ch}(\alpha + \alpha') + 1,$$ (26) $$Z_2 = z^2 - 2z \operatorname{ch}(\alpha - \alpha') + 1;$$

and with $\alpha > \alpha'$, the roots arranged in ascending order are

(27) $$-e^{\alpha+\alpha'}, \quad -e^{-\alpha-\alpha'}, \quad e^{-\alpha+\alpha'}, \quad e^{\alpha-\alpha'};$$

and denoting then by z_0, z_3, z_2, z_1,

(28) $$Z_1 = (z - z_0)(z - z_3),$$ (29) $$Z_2 = (z_2 - z)(z_1 - z),$$
and

(30) $$z_0 < -1 < z_3 < z < z_2 < 1 < z_1,$$

With $a = -1$, putting $\lambda, \lambda' = \operatorname{ch} \alpha, \operatorname{ch} \alpha'$,

(31) $$Z_1 = z_2 - 2z \operatorname{ch}(\alpha - \alpha') + 1, \quad Z_2 = -z^2 + 2z \operatorname{ch}(\alpha + \alpha') - 1,$$

and arranged in ascending order

(32) $$e^{-\alpha-\alpha'}, \quad e^{-\alpha+\alpha'}, \quad e^{\alpha-\alpha'}, \quad e^{\alpha+\alpha'},$$
denoted by z_3, z_2, z_1, z_0,

(33) $$Z_1 = (z_1 - z)(z_2 - z),$$ (34) $$Z_2 = (z_0 - z)(z - z_3),$$

(35) $$-1 < z_3 < z < z_2 < 1 < z_1 < z_0.$$

With $a = 1$, we can make

(36) $$A_1^2 Z_1 + A_2^2 Z_2 = \left(\frac{z \pm 1}{2}\right)^2,$$

by taking

(37) $$A_1 = \frac{\operatorname{sh} \frac{1}{2}(\alpha + \alpha') \text{ or } \operatorname{ch} \frac{1}{2}(\alpha + \alpha')}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}},$$ (38) $$A_2 = \frac{\operatorname{ch} \frac{1}{2}(\alpha - \alpha') \text{ or } \operatorname{sh} \frac{1}{2}(\alpha - \alpha')}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}},$$

and then put

(39) $$\alpha_1 = \left[\frac{\operatorname{sh} \frac{1}{2}(\alpha + \alpha')\sqrt{Z_1} + i \operatorname{ch} \frac{1}{2}(\alpha - \alpha')\sqrt{Z_2}}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}}\right]^{\frac{1}{2}},$$

(40) $$\beta_1 = \left[\frac{\operatorname{ch} \frac{1}{2}(\alpha + \alpha')\sqrt{Z_1} + i \operatorname{sh} \frac{1}{2}(\alpha - \alpha')\sqrt{Z_2}}{2\sqrt{\operatorname{ch} \alpha \operatorname{ch} \alpha'}}\right]^{\frac{1}{2}},$$

Then Klein's $\alpha, \beta, \gamma, \delta$, are of the form

(41) $$\alpha = e^{l_1 nti} G^{\frac{1}{2}\beta i} \alpha_1,$$ (42) $$\beta = e^{l_1' nti} G^{-\frac{1}{2}\beta i} \beta_1,$$ (43) $$\gamma = e^{-l_1' nti} G^{\frac{1}{2}\beta i} \gamma_1,$$

(44) $$\delta = e^{-l_1 nti} G^{-\frac{1}{2}\beta i} \delta_1;$$

and

(45) $\frac{1}{2} i \sin \vartheta e^{\psi i} = \alpha\beta = e^{(l_1 + l_2) nti} \alpha_1 \beta_1$,

(46) $\wp v' = e_1 = \frac{2}{3} \operatorname{sh}^2 \alpha - \frac{1}{3} \operatorname{sh}^2 \alpha' + \frac{1}{3}$,

(47) $\wp v = e_2 = - \frac{1}{3} \operatorname{sh}^2 \alpha + \frac{2}{3} \operatorname{sh}^2 \alpha' + \frac{1}{3}$,

(48) $e_3 = - \frac{1}{3} \operatorname{sh}^2 \alpha - \frac{1}{3} \operatorname{sh}^2 \alpha' - \frac{2}{3}$,

(49) $e_1 - e_2 = \operatorname{sh}^2 \alpha - \operatorname{sh}^2 \alpha'$, (50) $e_1 - e_3 = \operatorname{ch}^2 \alpha$, (51) $e_2 - e_3 = \operatorname{ch}^2 \alpha'$,

(52) $\wp 2w = - c + \frac{b^2}{a} = \frac{2}{3} \operatorname{sh}^2 \alpha + \frac{2}{3} \operatorname{sh}^2 \alpha' + \frac{1}{3} + \operatorname{sh}^2 \alpha \operatorname{sh}^2 \alpha'$,

(53) $\wp 2w - e_1 = \operatorname{ch}^2 \alpha \operatorname{sh}^2 \alpha'$, (54) $\wp 2w - e_2 = \operatorname{sh}^2 \alpha \operatorname{ch}^2 \alpha'$, (55) $\wp 2w - e_3 = \operatorname{ch}^2 \alpha \operatorname{ch}^2 \alpha'$,

(56) $\wp w - e_1 = \operatorname{sh}(\alpha + \alpha') \operatorname{ch} \alpha (\operatorname{ch} \alpha' + \operatorname{sh} \alpha')$,

(57) $\wp w - e_2 = \operatorname{sh}(\alpha + \alpha') \operatorname{ch} \alpha' (\operatorname{ch} \alpha + \operatorname{sh} \alpha)$,

(58) $\wp w - e_3 = \operatorname{ch} \alpha \operatorname{ch} \alpha' [\operatorname{ch}(\alpha + \alpha') + \operatorname{sh}(\alpha + \alpha')]$.

Try

(59) $L = L', \quad a = a', \quad D = E = - 4a \frac{B}{M}, \quad BM = - aL^2$,

(60) $Z = a(z - 1)^2 \left[(z + 1)^2 + 4a \frac{L^2}{M^2} z \right]$.

representing a state of steady motion.

Take $a = + 1$,

(61) $Z_1 = (z - 1)^2, \quad Z_2 = z^2 + 2z \operatorname{ch} 2\alpha + 1$,

(62) $\begin{matrix} \alpha_1 \\ \delta_1 \end{matrix} = \left[\frac{\operatorname{sh} \alpha (z - 1) \pm i \sqrt{Z_2}}{2 \operatorname{ch} \alpha} \right]^{\frac{1}{2}}, \quad \alpha_1 \delta_1 = \frac{z + 1}{2}$,

(63) $\beta_1 = \gamma_1 = \left(\frac{z - 1}{2} \right)^{\frac{1}{2}}, \quad \beta_1 \gamma_1 = \frac{z - 1}{2}, \quad \psi_1 = 0$.

Take $a = - 1$,

(64) $Z_1 = (z - 1)^2, \quad Z_2 = - z^2 + 2z \operatorname{ch} 2\alpha - 1$,

(65) $\begin{matrix} \alpha_1 \\ \delta_1 \end{matrix} = \left(\frac{\operatorname{ch} \alpha (z - 1) \pm i \sqrt{Z_2}}{2 \operatorname{sh} \alpha} \right)^{\frac{1}{2}}, \quad \alpha_1 \delta_1 = \frac{z + 1}{2}$,

(66) $\beta_1 = \gamma_1 = \left(\frac{z - 1}{2} \right)^{\frac{1}{2}}, \quad \beta_1 \gamma_1 = \frac{z - 1}{2}, \quad \psi_1 = 0$.

[*To be continued.*]

12. Before proceeding to the general resolution of the quartic Z, we can utilise the preceding analysis to the construction of some cases of algebraical motion, in the manner explained in the *American Journal of Mathematics*, XX, §33, by taking a parameter

$$v = \frac{2\omega_3}{\mu}, \quad \mu = 2n + 1, \tag{1}$$

and putting

$$\left\{\begin{aligned} \psi - pt &= \psi', \\ &= \frac{1}{2n+1} \cos^{-1} \frac{Hz^{2n+1} + H_1 z^{2n} + \dots}{(-z^2-1)^{n+\frac{1}{2}}} \\ &= \frac{1}{2n+1} \sin^{-1} \frac{Kz^{2n-1} + K_1 z^{2n-2} + \dots}{(-z^2+1)^{n+\frac{1}{2}}} \sqrt{Z}, \end{aligned}\right. \tag{2}$$

and then determining the various constants of the problem.

But the degree of the relation (2) can be halved by changing the variable from $z = \cos \vartheta$ to $t = \tan \frac{1}{2}\vartheta$, a different t from that representing time, henceforth to be distinguished when necessary by an accent, as t'.

This is equivalent to a change to the stereographic projection; and now with

$$z = \frac{1 - t^2}{1 + t^2}, \tag{3}$$

we can put

$$\left\{\begin{aligned} \psi' &= \frac{1}{n+\frac{1}{2}} \cos^{-1} \frac{ht^{2n-1} + h_1 t^{2n-2} + \dots}{t^{n+\frac{1}{2}}} \sqrt{T_1} \\ &= \frac{1}{n+\frac{1}{2}} \sin^{-1} \frac{ht^{2n-1} - h_1 t^{2n-2} + \dots}{t^{n+\frac{1}{2}}} \sqrt{T_2}, \end{aligned}\right. \tag{4}$$

making

$$\frac{d\psi'}{dt} = -\frac{(L + L')t^4 + 2Pt^2 + L - L'}{t\sqrt{T_1 T_2}}, \tag{5}$$

where

$$\left\{\begin{aligned} T_1 T_2 &= \tfrac{1}{4}(t^2 + 1)^4 M^2 Z = -(t^2 + 1)^2 \{(L + L')t^2 + L - L'\}^2, \\ &\quad + M^2 t^2 \left[-a(t^2 - 1)^2 - 4\frac{aB}{M}(t^4 - 1) + (a + D)(t^2 + 1)^2\right] \\ &= -(L + L')^2 t^8 + 4bt^6 + 6ct^4 + 4dt^2 - (L - L')^2, \end{aligned}\right. \tag{6}$$

suppose; and the troublesome determination of the homogeneity factor M is delayed, and does not hinder us.

To obtain the factors T_1 and T_2, write

(7) $$T_1T_2 = -[(L+L')t^4 + 2\gamma t^2 + L - L']^2 + 4t^2(\beta t^2 + \delta)^2,$$

so that

(8) $$T_1 = (L+L')t^4 + 2\beta t^3 + 2\gamma t^2 + 2\delta t + L - L',$$

(9) $$T_2 = -(L+L')t^4 + 2\beta t^3 - 2\gamma t^2 + 2\delta t - L + L',$$

and

(10) $$b = \beta^2 - (L+L')\gamma, \quad d = \delta^2 - (L-L')\gamma, \quad 6c = 8\beta\delta - 4\gamma^2 - 2(L^2 - L'^2).$$

Denoting the elliptic argument of the motion by u, such that

(11) $$u = \int \frac{dt^2}{\surd(T_1T_2)},$$

and making use of the Biermann-Weierstrass formula (2), §7, with the argument v_1 and v_2 corresponding now with $t^2 = 0$ and ∞, we find for $v_1 + v_2 = v$,

(12) $$12\wp v = 6c - 6(L^2 - L'^2) = 8\beta\delta - 4\gamma^2 - 8(L^2 - L'^2).$$

The formula

(13) $$\begin{cases} \wp'(v_1+v_2) = -\dfrac{(ay^3+3by^2+3cy+d)x + by^3 + 3cy^2 + 3dy + e}{(x-y)^3}\surd X \\ \qquad\qquad + \dfrac{(ax^3+3bx^2+3cx+d)y + bx^3 + 3cx^2 + 3dx + e}{(x-y)^3}\surd Y, \end{cases}$$

with x corresponding to $t^2 = 0$, and y to $t^2 = \infty$, leads to

(14) $$\begin{cases} i\wp' v = -(L-L')b - (L+L')d \\ \qquad = -(L-L')\beta^2 - (L+L')\delta^2 + 2(L^2 - L'^2)\gamma. \end{cases}$$

The quadrinvariant

(15) $$\begin{cases} g_2 = ae - 4bd + 3c^2 \\ \quad = (L^2 - L'^2)^2 - 4\{\beta^2 - (L+L')\gamma\}\{\delta^2 - (L-L')\gamma\} \\ \qquad\qquad + \tfrac{1}{12}\{8\beta\delta - 4\gamma^2 - 2(L^2 - L'^2)\}^2, \end{cases}$$

so that

(16) $$2\wp'' v = 12\wp^2 v - g_2 = 4(\beta\delta - L^2 + L'^2)^2 + 4\gamma i\wp' v = (\gamma^2 + 3\wp v)^2 + 4\gamma i\wp' v,$$

a quartic equation for the determination of γ.

A root of this equation is

(17) $$\gamma = -\tfrac{1}{2}\frac{\wp''\tfrac{1}{2}v}{i\wp'\tfrac{1}{2}v}, \quad \text{the others being} -\tfrac{1}{2}\frac{\wp''(\tfrac{1}{2}v + \omega)}{i\wp'(\tfrac{1}{2}v + \omega)};$$

so that, taking

(18) $$\tfrac{1}{2}v = \frac{2\omega_3}{2n+1}, \quad v = \frac{4\omega_3}{2n+1},$$

and employing Halphen's x and y, we can put (*L. M. S.*, XXV, p. 198)

$$(19) \qquad 12\wp v = -(y+1)^2 + 8x, \qquad i\wp' v = xy, \qquad \wp'' v = 2x^2 - xy(y+1);$$

$$(20) \qquad 12\wp \tfrac{1}{2} v = -(y+1)^2 - 4x, \qquad i\wp' \tfrac{1}{2} v = x, \qquad \wp'' \tfrac{1}{2} v = x(y+1),$$

and then

$$(21) \qquad \gamma = -\tfrac{1}{2}(y+1),$$

$$(22) \qquad \gamma^2 + 3\wp v = 2x,$$

$$(23) \qquad \beta\delta = L^2 - L'^2 + x = s + x,$$

putting

$$(24) \qquad L^2 - L'^2 = s, \text{ i. e. } s - \sigma, \text{ with } \sigma = 0.$$

$$(25) \qquad \left\{ \begin{aligned} (L - L')\beta^2 + (L + L')\delta^2 &= -(y+1)(L^2 - L'^2) - xy \\ &= -(y+1)s - xy, \end{aligned} \right.$$

$$(26) \qquad \{\sqrt{(L-L')}\beta \pm \sqrt{(L+L')}\delta\}^2 = \pm 2s^{\frac{1}{2}}(s+x) - (y+1)s - xy = S_1 \text{ or } S_2,$$

where

$$(27) \qquad S_1 S_2 = -4s(s+x)^2 + \{(y+1)s + xy\}^2 = -S;$$

and then

$$(28) \qquad \beta = \frac{\sqrt{S_1} + \sqrt{S_2}}{2\sqrt{(L-L')}}, \qquad \delta = \frac{\sqrt{S_1} - \sqrt{S_2}}{2\sqrt{(L'+L)}}.$$

With s negative and

$$(29) \qquad L'^2 - L^2 = -s = p^2,$$

$$(30) \qquad -(L'-L)\beta^2 + (L'+L)\delta^2 = (y+1)p^2 - xy,$$

$$(31) \qquad [(L'-L)\beta^2 + (L'+L)\delta^2]^2 = 4p^2(p^2 - x)^2 + [(y+1)p^2 - xy]^2 = p'^2,$$

suppose,

$$(32) \qquad \genfrac{}{}{0pt}{}{\beta}{\delta} = \sqrt{\frac{p' \mp [(y+1)p^2 - xy]}{2(L' \mp L)}}.$$

Try, as a special case,

$$(33) \qquad s + x = 0, \qquad \beta\delta = 0, \qquad \beta = 0 \text{ or } \sqrt{\frac{x}{L'-L}}, \qquad \delta = \sqrt{\frac{x}{L'+L}} \text{ or } 0,$$

$$(34) \qquad v' = \frac{2\omega_3}{\mu}, \quad v_1 = \frac{3\omega_3}{\mu}, \quad v_2 = \frac{\omega_3}{\mu},$$

so that it should be possible theoretically to express $\alpha_1, \beta_1, \gamma_1, \delta_1$, by the μth root of an algebraical function.

13. Returning to the original variable z by putting

$$t^2 = \frac{1-z}{1+z}, \tag{1}$$

$$\left\{\begin{aligned} M^2 Z &= -\tfrac{1}{4}[(L+L')(1-z)^2 + 2\gamma(1-z^2) + (L-L')(1+z^2)]^2 \\ &\qquad + (1-z^2)[\beta(1-z) + \delta(1+z)]^2 \\ &= -[2(L-L'z) - (L-\gamma)(1-z^2)]^2 \\ &\qquad + (1-z^2)[\beta + \delta - (\beta-\delta)z]^2, \end{aligned}\right. \tag{2}$$

$$\left\{\begin{aligned} M^2 Z + 4(L-L'z)^2 = (1-z^2)\{&4(L-\gamma)(L-L'z) - (L-\gamma)^2(1-z^2) \\ &+ (\beta+\delta)^2 - 2(\beta^2-\delta^2)z + (\beta-\delta)^2 z^2\}, \end{aligned}\right. \tag{3}$$

so that

$$-M^2 a = (\beta-\delta)^2 + (L-\gamma)^2, \tag{4}$$

implying $a = -1$ and an oblate figure for real values,

$$2aBM = -(\beta^2-\delta^2) - 2L'(L-\gamma), \tag{5}$$

$$M^2(a+D) = (\beta+\delta)^2 - (L-\gamma)^2 + 4L(L-\gamma). \tag{6}$$

Now with $a = -1$,

$$M^2 = L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x + \frac{L'\surd(-S) - Lxy}{s}, \tag{7}$$

$$2BM = 2LL' + \frac{L\surd(-S) - L'xy}{s}, \tag{8}$$

$$M^2(1-D) = -3L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x - \frac{L'\surd(-S) - Lxy}{s}, \tag{9}$$

$$M^2 D = 2\frac{L'\surd(-S) - Lxy}{s} + 4L^2, \tag{10}$$

$$M^2 E = 2\frac{L'\surd(-S) - xyL}{s} + 4L'^2, \tag{11}$$

$$\left\{\begin{aligned} M^2\rho^2 &= M^2\left(z^2 - 4\frac{B}{M}z - 1 + E\right) \\ &= [(\beta-\delta)z + \beta + \delta]^2 + [(L-\gamma)z + 2L']^2 - (L+\gamma)^2, \end{aligned}\right. \tag{12}$$

$$\frac{\Delta}{M^2} = L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x - \frac{Lxy}{s} - \frac{L'\surd(-S)}{s}, \tag{13}$$

$$\left\{\begin{aligned} \Delta &= \left[L^2 - 2s + \tfrac{1}{4}(y+1)^2 - 2x - \frac{Lxy}{s}\right]^2 + \frac{L'^2 S}{s^2} \\ &= \frac{-AL'^2 + B}{s}, \end{aligned}\right. \tag{14}$$

$$A = (L^2 + 3\wp v)^2 + 4Li\wp' v - 2\wp v'', \tag{15}$$

$$B = (L^3 + 3L\wp v + i\wp' v)^2. \tag{16}$$

Putting

(17) $L + \frac{1}{2}(y + 1) = \lambda,$

(18) $$A = \lambda\left(\lambda - \tfrac{1}{2}\sqrt{\frac{\sigma_2\sigma_3}{-\sigma_1}}\right)\left(\lambda - \tfrac{1}{2}\sqrt{\frac{-\sigma_1\sigma_3}{\sigma_2}}\right)\left(\lambda - \tfrac{1}{2}\sqrt{\frac{-\sigma_1\sigma_2}{\sigma_3}}\right),$$

(19) $$\sigma_a = s - s_a.$$

In Kirchhoff's special case, considered in the *American Journal of Mathematics*, XX, $B = 0$, and therefore

(20) $$\left(2LL' - \frac{L'xy}{s}\right)^2 + \frac{L^2S}{s^2} = 0,$$

(21) $$L'^2 = \frac{4L^2(L^2 + x)^2 - [(y + 1)L^2 + xy]^2}{4L[L^3 - \{\frac{1}{4}(y + 1)^2 - 2x\}L + xy]},$$

(22) $$LM^2 = -L^3 + \{\tfrac{1}{4}(y + 1)^2 - 2x\}L - xy,$$

(23) $$4L^2L'^2M^2 = -4L^2(L^2 + x)^2 + [(y + 1)L^2 + xy]^2,$$

(24) $$LL'M = aN_1N_2N_3,$$

as in the *American Journal of Mathematics*, XX (59).

14. We can illustrate the preceding theory immediately for the simplest values of μ, utilising the analysis in the *Phil. Trans.*, 1904.

(1) $$\mu = 3, \quad v = \tfrac{4}{3}\omega_3, \quad x = 0,$$

(2) $$\beta\delta = L^2 - L'^2, \quad \beta = (L + L')m, \quad \delta = \frac{L - L'}{m},$$

(3) $$T_1, T_2 = \pm(L + L')t^4 + 2(L + L')mt^3 \pm 2\gamma t^2 + 2\frac{L - L'}{m}t \pm (L - L'),$$

and then

(4) $$\psi = \tfrac{2}{3}\cos^{-1}\frac{t - m}{Nt^{\frac{3}{2}}}\sqrt{T_1} = \tfrac{2}{3}\sin^{-1}\frac{t + m}{Nt^{\frac{3}{2}}}\sqrt{T_2},$$

where

(5) $$N^2 = 4m\left[(L + L')m^2 - 2\gamma + \frac{L - L'}{m^2}\right],$$

leads on differentiation to

(6) $$\frac{d\psi}{dt} = -\frac{(L + L')t^4 + \frac{2}{3}\gamma t^2 + L - L'}{t\sqrt{(T_1T_2)}},$$

and so provides an algebraical case.

For higher values of $\mu = 2n + 1$, the work is rendered more symmetrical by the use of a new variable q, defined by

$$(7) \qquad t = q\sqrt[4]{\frac{L - L'}{L + L'}}, \text{ or } q\sqrt[4]{\frac{L' - L}{L' + L}},$$

according as s is positive $= p^2$, or negative $= -p^2$.

Now with s positive and

$$(8) \qquad \frac{T_1}{L - L'} = Q_1, \quad \frac{T_2}{L - L'} = Q_2,$$

$$(9) \qquad \begin{matrix} Q_1 \\ Q_2 \end{matrix} = \pm q^4 + \frac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 \mp \frac{y + 1}{p} q^2 + \frac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q \pm 1,$$

$$(10) \qquad \begin{matrix} S_1 \\ S_2 \end{matrix} = \pm 2p(p^2 + x) - (y + 1)p^2 - xy,$$

we have to satisfy the relation

$$(11) \qquad \frac{d\psi}{dq} = -\frac{q^4 + 2q^2 \frac{P}{p} + 1}{q\sqrt{Q_1 Q_2}},$$

by an expression of the form

$$(12) \qquad \begin{cases} \psi = \dfrac{1}{n + \frac{1}{2}} \cos^{-1} \dfrac{Aq^{2n-1} + Bq^{2n-2} + \ldots\ldots + Kq + L}{Nq^{n+\frac{1}{2}}} \sqrt{Q_1}, \\ \quad = \dfrac{1}{n + \frac{1}{2}} \sin^{-1} \dfrac{Aq^{2n-1} - Bq^{2n-2} + \ldots\ldots + Kq - L}{Nq^{n+\frac{1}{2}}} \sqrt{Q_2}, \end{cases}$$

so that, differentiating, the coefficients $A, B, \ldots\ldots, K, L$ (the two L's must be kept distinct), are obtained from the identity

$$(13) \qquad \begin{cases} [(2n - 3)Aq^{2n-1} + (2n - 5)Bq^{2n-2} + \ldots\ldots - (2n - 1)Kq - (2n + 1)L] \\ \qquad \left(q^4 + \dfrac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 - \dfrac{y + 1}{p} q^2 + \dfrac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q + 1\right) \\ + (Aq^{2n-1} + Bq^{2n-2} + \ldots\ldots + Kq + L) \\ \qquad \left(4q^4 + 3\dfrac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 - 2\dfrac{y + 1}{p} q^2 + \dfrac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q + 0\right) \\ = (Aq^{2n-1} - Bq^{2n-2} + \ldots\ldots + Kq - L) \\ \qquad \left[(2n + 1)q^4 + 2q^2 \dfrac{(2n + 1)P}{p} + 2n + 1\right]. \end{cases}$$

Hence

$$\frac{B}{A} = -\frac{\surd S_1 + \surd S_2}{2p^{\frac{1}{2}}}, \quad \frac{K}{L} = -\frac{\surd S_1 - \surd S_2}{2p^{\frac{1}{2}}}, \tag{14}$$

$$\left\{\begin{aligned} \frac{C}{A} &= -(n-1)\frac{(\surd S_1 + \surd S_2)^2}{4p^3} - \frac{(2n-1)(y+1)+(4n+2)P}{4p} \\ \frac{H}{L} &= -(n-1)\frac{(\surd S_1 - \surd S_2)^2}{4p^3} - \frac{(2n-1)(y+1)+(4n+2)P}{4p}, \end{aligned}\right. \tag{15}$$

and so on.

Induction leads us to infer that

$$\frac{L}{A} = e^{(n+\frac{1}{2})I(v)} = \frac{E_1\surd S_1 + E_2\surd S_2}{2p^{n+\frac{1}{2}}}, \tag{16}$$

derived from the associated pseudo-elliptic integral

$$\left\{\begin{aligned} I(v) &= \int \frac{Ps - \frac{1}{2}xy}{s}\,\frac{ds}{\surd(-S)} \\ &= \frac{1}{n+\frac{1}{2}}\,\mathrm{ch}^{-1}\frac{E_1\surd S_1}{2p^{n+\frac{1}{2}}} = \frac{1}{n+\frac{1}{2}}\,\mathrm{sh}^{-1}\frac{E_2\surd S_2}{2p^{n+\frac{1}{2}}}, \end{aligned}\right. \tag{17}$$

so that we may take A and L as reciprocal, and $AL = -1$,

$$\frac{A}{L} = \pm\sqrt{\frac{E_1\surd S_1 \mp E_2\surd S_2}{2p^{n+\frac{1}{2}}}}, \tag{18}$$

and thence $\frac{B}{K}, \frac{C}{H}, \dots\dots$

In the second case of s negative we have to change p^2 into $-p^2$, and q^4 into $-q^4$, and take

$$\frac{T_1}{L'-L} = Q_1, \quad \frac{T_2}{L'-L} = Q_2, \tag{19}$$

$$\begin{matrix} Q_1 \\ Q_2 \end{matrix} = \pm q^4 + q^3\sqrt{\frac{p'+R}{2p^{\frac{1}{2}}}} \mp q^2\frac{y+1}{p} + q\sqrt{\frac{p'-R}{2p^{\frac{1}{2}}}} \mp 1, \tag{20}$$

where

$$p'^2 = 4p^2(p^2-x)^2 + [(y+1)p^2 - xy]^2, \quad R = -(y+1)p^2 + xy. \tag{21}$$

Tested in this way for $\mu = 3$,

$$\left\{\begin{aligned} \psi &= \tfrac{1}{3}\cos^{-1}\frac{(Aq+L)\surd Q_1}{Nq^{\frac{3}{2}}} \\ &= \tfrac{1}{3}\sin^{-1}\frac{(Aq-L)\surd Q_2}{Nq^{\frac{3}{2}}}, \end{aligned}\right. \tag{22}$$

with

(23) $$\begin{matrix} Q_1 \\ Q_2 \end{matrix} = \pm q^4 + \frac{\sqrt{S_1} + \sqrt{S_2}}{p^{\frac{3}{2}}} q^3 \pm \frac{q^2}{p} + \frac{\sqrt{S_1} - \sqrt{S_2}}{p^{\frac{3}{2}}} q \pm 1,$$

(24) $$\begin{matrix} S_1 \\ S_2 \end{matrix} = \pm 2p^3 - p^2 + c,$$

(25) $$\begin{matrix} A \\ L \end{matrix} = \pm \sqrt{\frac{\sqrt{S_1} \mp \sqrt{S_2}}{2p^{\frac{3}{2}}}},$$

(26) $$N^2 = \frac{4c}{p^3},$$

leading to

(27) $$\frac{d\psi}{dq} = - \frac{q^4 + \frac{q^2}{3p} + 1}{q\sqrt{Q_1 Q_2}}.$$

Tested by $\mu = 5$, we find for

(28) $$v = \tfrac{4}{5}\omega_3, \quad x = y, \quad P = \frac{1 - 3x}{10},$$

(29) $$\begin{matrix} S_1 \\ S_2 \end{matrix} = \pm 2p^3 - (1 + y)p^2 \pm 2yp - y^2,$$

(30) $$\begin{matrix} A \\ L \end{matrix} = \pm \sqrt{\frac{(p + y)\sqrt{S_1} \mp (-p + y)\sqrt{S_2}}{2p^{\frac{5}{2}}}}, \quad AL = -1,$$

(31) $$\begin{matrix} B \\ K \end{matrix} = \mp \sqrt{\frac{-(p^2 + p + y)\sqrt{S_1} \pm (p^2 - p + y)\sqrt{S_2}}{2p^{\frac{5}{2}}}},$$

(32) $$\left\{\begin{aligned} \psi &= \tfrac{2}{5} \cos^{-1} \frac{Aq^3 + Bq^2 + Kq + L}{Nq^{\frac{5}{2}}} \sqrt{Q_1}, \\ &= \tfrac{2}{5} \sin^{-1} \frac{Aq^3 - Bq^2 + Kq - L}{Nq^{\frac{5}{2}}} \sqrt{Q_2}, \end{aligned}\right.$$

(33) $$N^2 = \frac{4y^3}{p^5}.$$

This can also be written

(34) $$\left\{\begin{aligned} \psi &= \tfrac{2}{5} \cos^{-1} \frac{q^3 + B'q^2 + K'q + L'}{N'q^{\frac{5}{2}}} \sqrt{Q_1} \\ &= \tfrac{2}{5} \sin^{-1} \frac{-q^3 + B'q^2 - K'q + L'}{N'q^{\frac{5}{2}}} \sqrt{Q_2}, \end{aligned}\right.$$

(35) $$B' = \frac{B}{A} = - \frac{\sqrt{S_1} + \sqrt{S_2}}{2p^{\frac{3}{2}}},$$

(36) $$K' = \frac{K}{A} = \frac{(-1 + y)p^2 + y^2 - \sqrt{S_1}\sqrt{S_2}}{2p^3},$$

(37) $$L' = \frac{L}{A} = - \frac{(p + y)\sqrt{S_1} + (-p + y)\sqrt{S_2}}{2p^{\frac{5}{2}}}.$$

Tested by $\mu = 7$ (*Phil. Trans.*, §9),

$$(38)\quad v = \tfrac{4}{7}\omega_3, \quad x = z(1-z)^2, \quad y = z(1-z), \quad P = \frac{3 - 9z + 5z^2}{14},$$

$$(39)\begin{cases} \dfrac{A}{L} = \pm e^{\mp\frac{7}{2}iI(v)} \\ \quad = \pm\sqrt{\dfrac{[p^2+(1-z)^2p+z(1-z)^3]\surd S_1 \mp [p^2-(1-z)^2p+z(1-z)^3]\surd S_2}{2p^{\frac{7}{2}}}}, \end{cases}$$

and thence

$$(40)\quad \frac{B}{K} = -\sqrt{\frac{\begin{array}{c}[p^3-(1-z)^2p^2-(1-z)^3p-z(1-z)^4]\surd S_1 \\ \mp[-p^3-(1-z)^2p^2+(1-z)^3p-z(1-z)^4]\surd S_2\end{array}}{2p^{\frac{9}{2}}}}$$

$$(41)\quad \frac{C}{H} = -\sqrt{\frac{\begin{array}{c}[-4p^4+0+(5z^2-4z^3)p^2+z^2(1-z)^2p+z^3(1-z)^3]\surd S_1 \\ \mp[-4p^4+0+(5z^2-4z^3)p^2-z^2(1-z)^2p+z^3(1-z)^3]\surd S_2\end{array}}{2p^{\frac{11}{2}}}}$$

in

$$(42)\begin{cases} \psi' = \tfrac{2}{7}\cos^{-1}\dfrac{Aq^5+Bq^4+Cq^3+Hq^2+Kq+L}{Nq^{\frac{7}{2}}}\surd Q_1 \\ \quad = \tfrac{2}{7}\sin^{-1}\dfrac{Aq^5-Bq^4+Cq^3-Hq^2+Kq-L}{Nq^{\frac{7}{2}}}\surd Q_2, \end{cases}$$

$$(43)\quad N^2 = -\frac{4z^5(1-z)^6}{p^7}.$$

so that Q_1 and Q_2 must change sign when z is positive.

Similarly for $\mu = 9$, *Phil. Trans.*, §10,

$$(44)\begin{cases} v = \tfrac{4}{9}\omega_3, \quad x = c^3(1-c)(1-c+c^2), \quad y = c^2(1-c), \\ \qquad 18P = 1 + 0 - 3c^2 + 7c^3 \end{cases}$$

$$(45)\quad \frac{A}{L} = \mp\sqrt{\frac{(p^3+h_1p^2+h_2p+h_3)\surd S_1 \mp (-p^3+h_1p^2-h_2p+h_3)\surd S_2}{2p^{\frac{9}{2}}}},$$

$$(46)\quad h_1 = c^2(1-2c), \quad h_2 = -c^4(1-c+c^2), \quad h_3 = -c^6(1-c)(1-c+c^2);$$

and thence the remaining coefficients

$$(47)\quad \frac{B}{K} = \mp\sqrt{\frac{\begin{array}{c}(-p^4+k_1p^3+k_2p^2+k_3p+k_4)\surd S_1 \\ \mp(-p^4-k_1p^3+k_2p^2-k_3p+k_4)\surd S_2,\end{array}}{2p^{\frac{11}{2}}}}$$

$$(48)\quad \begin{cases} k_1 = -(1+c^3), \quad k_2 = -c^2(1-c+c^2)(1-c-c^2), \\ \qquad k_3 = c^4(1-c+c^2)^2, \quad k_4 = c^6(1-c)(1-c+c^2)^2; \end{cases}$$

and so on, in

$$(49)\quad \begin{cases} \psi' = \frac{2}{9}\cos^{-1}(Aq^7 + Bq^6 + Cq^5 + Dq^4 + Gq^3 + Hq^2 + Kq + L)\surd Q_1 \div Nq^{\frac{9}{2}} \\ \quad = \frac{2}{9}\sin^{-1}(Aq^7 - Bq^6 + Cq^5 + Dq^4 + Gq^3 - Hq^2 + Kq - L)\surd Q_2 \div Nq^{\frac{9}{2}}, \end{cases}$$

$$(50)\qquad N = \frac{2c^2(1-c)^2(1-c+c^2)}{p^{\frac{9}{2}}}.$$

Material for the construction of the case of $\mu = 11, 13, 15, 17, \ldots.,$ will be found in *Phil. Trans.*, §§11, 12, 13, 15, ...

15. With a parameter v a fraction of the imaginary period ω_3 we must take $a = -1$, as stated above in (4), §13; this is also evident from the graph of the quartic Z, which must now have all its roots included between $z = -1$ and $+1$.

At the same time as ψ' is given in (2), §12, $\varpi' = \varpi - pt$ is given by a relation of similar form,

$$(1)\quad \begin{cases} \varpi' = \frac{1}{2n+1}\cos^{-1}\frac{Hz^{2n+1} + H_1z^{2n} + \ldots.}{\rho^{2n+1}} \\ \quad = \frac{1}{2n+1}\sin^{-1}\frac{Kz^{2n-1} + K_1z^{2n-2} + \ldots.}{\rho^{2n+1}}\surd Z, \end{cases}$$

where, as in (7), §5,

$$(2)\qquad \rho^2 = \left(z - 2\frac{B}{M}\right)^2 - k, \quad k = 1 + aE + 4\frac{B^2}{M^2}.$$

If k is positive, denote it by λ^2, and put

$$(3)\qquad z - 2\frac{B}{M} = \lambda\frac{v^2+1}{v^2-1}, \quad dz = -\frac{4\lambda v dv}{(v^2-1)^2},$$

$$(4)\qquad \rho = \lambda\frac{2v}{v^2-1},$$

where this new v must not be confused with the parameter v.

The degree can now be halved, as before in §12 for ψ', so that

$$(5)\quad \begin{cases} \varpi' = \frac{1}{n+\frac{1}{2}}\cos^{-1}\frac{kv^{2n-1} + k_1v^{2n-2} + \ldots.}{Nv^{n+\frac{1}{2}}}\surd V_1 \\ \quad = \frac{1}{n+\frac{1}{2}}\sin^{-1}\frac{kv^{2n-1} - k_1v^{2n-2} + \ldots.}{Nv^{n+\frac{1}{2}}}\surd V_2, \end{cases}$$

which is required to satisfy the differential relation, from (12), §4,

(6)
$$\left\{\begin{aligned} d\varpi' &= \left[2\frac{\left(z-2\frac{B}{M}\right)\left(\frac{Lz-L'}{M}\right)}{\left(z-2\frac{B}{M}\right)^2-k} - \frac{L-P}{M}\right]\frac{dz}{\sqrt{Z}} \\ &= \frac{\frac{L+P}{M}\left(z-2\frac{B}{M}\right)^2 + 2\frac{L''\lambda}{M}\left(z-2\frac{B}{M}\right) + \frac{L-P}{M}\lambda^2}{\left(z-2\frac{B}{M}\right)^2-k}\frac{dz}{\sqrt{Z}}, \end{aligned}\right.$$

where

(7)
$$2\frac{BL}{M} - L' = L''\lambda;$$

and thence

(8)
$$\frac{d\varpi'}{dv} = -\frac{(L+L'')v^4 + 2Pv^2 + L - L''}{v\sqrt{(V_1V_2)}},$$

where

(9)
$$M^2Z = \frac{4\lambda^2 V_1 V_2}{(v^2-1)^4},$$

a relation similar to (6), §12.

Writing the quartic Z in the form (3), §3, we saw in (14), §7, that the elliptic arguments v_3, v_4 corresponding to

(10)
$$\rho = 0, \quad z = 2\frac{B}{M} \pm \lambda,$$

were such that

(11)
$$v_3 + v_4 = v = v_1 + v_2.$$

But written in the form

(12)
$$Z = a(z^2 - F)\left[\left(z - 2\frac{B}{M}\right)^2 - \lambda^2\right] - 4(Hz - K)^2,$$

the values $z = \pm\sqrt{F}$ correspond to arguments v_5 and v_6, such that

(13)
$$v_5 - v_6 = v_3 - v_4.$$

Equating the coefficients in the two forms of Z,

(14)
$$4H^2 = -a(F-1) + 4\frac{L^2}{M^2},$$

(15)
$$4HK = -2a(F-1)\frac{B}{M} + 4\frac{LL'}{M^2},$$

(16)
$$4K^2 = a(F-1)\left(4\frac{B^2}{M^2} - k\right) + 4\frac{L'^2}{M^2};$$

and eliminating H and K, we obtain the equation for F, such that, rejecting $F-1=0$,

(17) $$F-1=4a\frac{L^2-L''^2}{M^2},$$

(18) $$Hz-K=\frac{L''}{M}\left(z-2\frac{B}{M}\right)+\frac{L\lambda}{M}.$$

Then

(19) $$Z+4\left[\frac{L''}{M}\left(z-2\frac{B}{M}\right)+\frac{L\lambda}{M}\right]^2=a(z^2-F)\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right],$$

and introducing an arbitrary α,

(20) $$\left\{\begin{aligned}&\lambda^2Z+\left\{\alpha\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right]-2\frac{L''\lambda}{M}\left(z-2\frac{B}{M}\right)-\frac{2L\lambda^2}{M}\right\}^2\\&\quad=\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right]\left\{\alpha^2\left[\left(z-2\frac{B}{M}\right)^2-\lambda^2\right]-4\lambda\alpha\left[\frac{L''}{M}\left(z-2\frac{B}{M}\right)+\frac{L\lambda}{M}\right]\right.\\&\quad\left.+\lambda^2a\left[\left(z-2\frac{B}{M}\right)^2+4\frac{B}{M}\left(z-2\frac{B}{M}\right)+4\frac{B^2}{M^2}-F\right]\right\}.\end{aligned}\right.$$

Taking the second quadratic factor

(21) $$\left\{\begin{aligned}&(\alpha^2+a\lambda^2)\left(z-2\frac{B}{M}\right)^2-4\lambda\left(\frac{L''}{M}\alpha-\frac{aB\lambda}{M}\right)\left(z-2\frac{B}{M}\right)\\&\qquad\qquad-\left(\alpha^2+4\frac{L}{M}\alpha-4\frac{aB}{M^2}+aF\right)^2,\end{aligned}\right.$$

make it a square by the condition

(22) $$(\alpha^2+a\lambda^2)\left(\alpha^2+4\frac{L}{M}\alpha-4\frac{aB^2}{M^2}+aF\right)+4\left(\frac{L''}{M}\alpha-\frac{aB\lambda}{M}\right)^2=0.$$

This quartic equation on putting

(23) $$\alpha=\frac{\gamma-L}{M},$$

will be found to reduce to

(24) $$(\gamma^2+3\wp v)^2+4\gamma i\wp' v-2\wp'' v=0,$$

as before in (16), §12.

Then we can write

(25) $$\left\{\begin{aligned}&V_1V_2=\tfrac{1}{4}\frac{M^2}{\lambda^2}Z(v^2-1)^4=-[(L+L'')v^4-(1+y)v^2+L-L'']^2\\&\qquad\qquad+4v^2(\beta'v^2+\delta')^2,\end{aligned}\right.$$

of the same form as $T_1 T_2$ in (7), §12, and the resolution is effected in the same way, so that the preceding algebraical results for ψ can be utilised with slight modification for ϖ'.

When k is negative we must take

$$z - 2\frac{B}{M} = \lambda \frac{-w^2+1}{2w}, \quad \lambda^2 = -k, \tag{26}$$

$$\rho = \lambda \frac{w^2+1}{2w}, \tag{27}$$

to effect the halving of the degree of the algebraical results in the form

$$\left\{ \begin{aligned} \varpi' &= \frac{1}{n+\frac{1}{2}} \cos^{-1} \frac{Hw^{2n-1} + \ldots\ldots}{(w^2+1)^{n+\frac{1}{2}}} \surd W_1, \\ &= \frac{1}{n+\frac{1}{2}} \sin^{-1} \frac{Kw^{2n-1} + \ldots\ldots}{(w^2+1)^{n+\frac{1}{2}}} \surd W_2, \end{aligned} \right. \tag{28}$$

satisfying the differential relation

$$\frac{d\varpi'}{dw} = -2\frac{(L+P)(w^2+1)^2 - 4L''(w^3-w) - 8Lw^3}{(w^2+1)\surd(W_1 W_2)}. \tag{29}$$

We can however infer the result from the form when k is positive by putting

$$v = \frac{w+1}{-w+1}, \quad k = \lambda^2, \tag{30}$$

$$z - 2\frac{B}{M} = \lambda \frac{v^2+1}{v^2-1} = \lambda \frac{w^2+1}{2w}, \tag{31}$$

$$\rho = \lambda \frac{2v}{v^2-1} = \lambda \frac{-w^2+1}{2w}, \tag{32}$$

when

$$\frac{d\varpi'}{dw} = -2\frac{(L+P)(w^2-1)^2 + 4L''(w^3+w) + 8Lw^2}{(-w^2+1)\surd(W_1 W_2)}, \tag{33}$$

and then a change of λ into λi, L'' into $L''i$, and w into wi, with appropriate change in the coefficients will enable us to pass from positive to negative k.

Now with positive k, and

$$V_1 = \alpha v^4 + 2\beta v^3 + 2\gamma v^2 + 2\delta v + \epsilon, \tag{34}$$

$$V_2 = -\alpha v^4 + 2\beta v^3 - 2\gamma v^2 + 2\delta v - \epsilon, \tag{35}$$

and with

(36) $$\begin{cases} W_1 = (w-1)^4 V_1 \\ \quad = aw^4 + 4bw^3 + 6cw^2 + 4dw + e, \end{cases}$$

we find

(37) $$W_2 = -ew^4 - 4dw^3 - 6cw^2 - 4dw - e.$$

16. In the algebraical case

(1) $$\begin{cases} \varpi' = \dfrac{1}{n+\frac{1}{2}} \cos^{-1} \dfrac{H_0 w^{2n-1} + H_1 w^{2n-2} + \ldots + H_{2n-1}}{(-w^2+1)^{n+\frac{1}{2}}} \surd W_1 \\ \quad = \dfrac{1}{n+\frac{1}{2}} \sin^{-1} \dfrac{H_{2n-1} w^{2n-1} + H_{2n-2} w^{2n-2} + \ldots + H_0}{(-w^2+1)^{n+\frac{1}{2}}} \surd W_2. \end{cases}$$

Thus for instance, for $\mu = 3$,

(2) $$\varpi' = \tfrac{2}{3} \cos^{-1} \frac{v-n}{Nv^{\frac{3}{2}}} \surd V_1 = \tfrac{2}{3} \sin^{-1} \frac{v+n}{Nv^{\frac{3}{2}}} \surd V_2,$$

(3) $$V_1, V_2 = \pm(L+L'')v^4 + 2(L+L'')nv^3 \pm 2\gamma v^2 + 2(L-L'')\frac{v}{n} \pm (L-L'')$$

leading on differentiation to

(4) $$\frac{d\varpi'}{dv} = -\frac{(L+L'')v^4 + \frac{2}{3}\gamma v^2 + L - L''}{v\surd(V_1 V_2)};$$

and replacing v by the new variable w from (30), §15,

(5) $$\varpi' = \tfrac{2}{3} \cos^{-1} \frac{Aw+B}{N(-w^2+1)^{\frac{3}{2}}} \surd W_1 = \tfrac{2}{3} \sin^{-1} \frac{Bw+A}{N(-w^2+1)^{\frac{3}{2}}} \surd W_2,$$

where

(6) $A = 1 + n, \quad B = 1 - n,$

(7) $W_1 = aw^4 + 4bw^3 + 6cw^2 + 4dw + e,$

(8) $W_2 = -ew^4 - 4dw^3 - 6cw^2 - 4bw - a,$

(9) $a = 2(L+\gamma) - 2(L+L'')n - 2(L-L'')\dfrac{1}{n},$

(10) $e = 2(L+\gamma) + 2(L+L'')n + 2(L-L'')\dfrac{1}{n},$

(11) $b = 2L'' - (L+L'')n + (L-L'')\dfrac{1}{n},$

(12) $d = 2L'' + (L+L'')n - (L-L'')\dfrac{1}{n};$

(13) $c = 2L - \tfrac{2}{3}\gamma.$

Putting $n = e^a$, we can write

$$\varpi' = \tfrac{2}{3}\cos^{-1}\frac{w\,\mathrm{ch}\tfrac{1}{2}a - \mathrm{sh}\tfrac{1}{2}a}{N(-w^2+1)^{\frac{3}{2}}}\sqrt{W_1} = \tfrac{2}{3}\sin^{-1}\frac{w\,\mathrm{sh}\tfrac{1}{2}a - \mathrm{ch}\tfrac{1}{2}a}{N(-w^2+1)^{\frac{3}{2}}}\sqrt{W_2}, \tag{14}$$

$$a = L + \gamma - 2L\,\mathrm{ch}a - 2L''\mathrm{sh}a, \tag{15}$$

$$e = L + \gamma + 2L\,\mathrm{ch}a + 2L''\mathrm{sh}a, \tag{16}$$

$$b = L'' - L\,\mathrm{sh}a - L''\mathrm{ch}a, \tag{17}$$

$$d = L'' + L\,\mathrm{sh}a + L''\mathrm{ch}a, \tag{18}$$

$$c = L - \tfrac{1}{3}\gamma, \tag{19}$$

leading on differentiation to (33), §15.

Now to pass to negative k, replace w, λ, L'', a, by wi, λi, $L''i$, ai; so we replace ch$a$ by cos sh$a$, α by i sin a, and put

$$W_1 = aw^4 + 4bw^3 + 6cw^2 + 4dw + e \tag{20}$$

$$W_2 = ew^4 - 4dw^3 + 6cw^2 - 4bw + a, \tag{21}$$

with

$$a = L + \gamma - 2L\cos a - 2L''\sin a, \tag{22}$$

$$e = L + \gamma + 2L\cos a + 2L''\sin a, \tag{23}$$

$$b = -L'' - L\sin a + L''\cos a, \tag{24}$$

$$d = +L'' - L\sin a + L''\cos a, \tag{25}$$

$$c = -L + \tfrac{1}{3}\gamma; \tag{26}$$

and now

$$\varpi' = \tfrac{2}{3}\cos^{-1}\frac{w\cos\tfrac{1}{2}a - \sin\tfrac{1}{2}a}{N(w^2+1)^{\frac{3}{2}}}\sqrt{W_1} = \tfrac{2}{3}\sin^{-1}\frac{w\sin\tfrac{1}{2}a + \cos\tfrac{1}{2}a}{N(w^2+1)^{\frac{3}{2}}}\sqrt{W_2}, \tag{27}$$

satisfying the relation

$$(w\cos\tfrac{1}{2}a - \sin\tfrac{1}{2}a)^2W_1 + (w\sin\tfrac{1}{2}a + \cos\tfrac{1}{2}a)^2W_2 = N^2(w^2+1)^3, \tag{28}$$

$$N^2 = \gamma - L\cos 2a - L''\sin 2a; \tag{29}$$

$$\frac{d\varpi'}{dw} = -2\frac{(L+\tfrac{1}{3}\gamma)(w^2+1)^2 - 4L''w(w^2-1) - 8Lw^2}{(w^2+1)\sqrt{(W_1W_2)}}. \tag{30}$$

We can now infer that the general result in the equation (1), with $k = -\lambda^2$, assumes the form

$$\left\{\begin{aligned} \varpi' &= \frac{1}{n+\frac{1}{2}}\cos^{-1}\frac{H_0w^{2n-1} + H_1w^{2n-2} + \dots + H_{2n-1}}{(w^2+1)^{n+\frac{1}{2}}}\sqrt{W_1} \\ &= \frac{1}{n+\frac{1}{2}}\sin^{-1}\frac{H_{2n-1}w^{2n-1} - H_{2n-2}w^{2n-2} + \dots - H_0}{(w^2+1)^{n+\frac{1}{2}}}\sqrt{W_2}, \end{aligned}\right. \tag{31}$$

with W_1 and W_2 related as in (20) and (21).

The details of the calculation for $\mu = 2n + 1 = 5, 7, 9, \dots$, can be left as an exercise.

17. With μ even, $= 2n$, it is not possible to have

$$(1)\quad \begin{cases} \psi = \dfrac{1}{n}\cos^{-1}\dfrac{ht^{2n-2}+h_1t^{2n-3}+\dots\dots}{Nt^n}\sqrt{T_1} \\ \quad = \dfrac{1}{n}\sin^{-1}\dfrac{ht^{2n-2}-h_1t^{2n-3}+\dots\dots}{Nt^n}\sqrt{T_2}, \end{cases}$$

because, squaring and adding,

$$(2)\qquad (ht^{2n-2}+h_1t^{2n-3}+\dots\dots)^2T_1+(ht^{2n-2}-h_1t^{2n-3}+\dots\dots)^2T_2,$$

consists of odd powers of t, and cannot be made equal to N^2t^{2n}.

But in the case of μ even, T breaks up into factors quadratic in t^2, of the form

$$(3)\quad T_1 = \alpha_1t^4+2\beta_1t^2+\gamma_1, \qquad (4)\quad T_2 = -\alpha_2t^4-2\beta_2t^2-\gamma_2,$$

and from Laguerre's formula (16) §7, when

$$(5)\qquad v_1+v_2 = v = f\omega_3, \quad v_1-v_2 = v' = f'\omega_3,$$

$$(6)\qquad \sqrt{e_3-\wp v} = \sqrt{s_3-\sigma} = \frac{1}{2}\left(\sqrt{\alpha_2\gamma_1}+\sqrt{\alpha_1\gamma_2}\right),$$

$$(7)\qquad \sqrt{e_3-\wp v'} = \sqrt{s_3-s} = \frac{1}{2}\left(\sqrt{\alpha_2\gamma_1}-\sqrt{\alpha_1\gamma_2}\right),$$

$$(8)\qquad \sqrt{\alpha_2\gamma_1} = \sqrt{s_3-\sigma}+\sqrt{s_3-s}, \qquad (9)\quad \sqrt{\alpha_1\gamma_2} = \sqrt{s_3-\sigma}-\sqrt{s_3-s},$$

and

$$(10)\qquad \alpha_1\alpha_2 = (L+L')^2, \qquad (11)\quad \gamma_1\gamma_2 = (L-L')^2,$$

$$(12)\qquad \sqrt{\alpha_1\alpha_2\gamma_1\gamma_2} = L^2-L'^2 = s-\sigma = s(v')-s(v),$$

where σ or $s(v)$ was replaced by zero in §12 with μ odd.

Formula (13) §12 gives

$$(13)\quad \sqrt{4 . s_1-\sigma . s_2-\sigma . s_3-\sigma} = \tfrac{1}{2}(\alpha_1\beta_2+\alpha_2\beta_1)\sqrt{\gamma_1\gamma_2}+\tfrac{1}{2}(\beta_1\gamma_2+\beta_2\gamma_1)\sqrt{\alpha_1\alpha_2}$$
$$= \tfrac{1}{2}\left(\sqrt{\alpha_2\gamma_1}+\sqrt{\alpha_1\gamma_2}\right)\left(\beta_1\sqrt{\alpha_2\gamma_2}+\beta_2\sqrt{\alpha_1\gamma_1}\right),$$

so that, from (6),

$$(14)\qquad \sqrt{s_1-\sigma . s_2-\sigma} = \tfrac{1}{2}\left(\beta_1\sqrt{\alpha_2\gamma_2}+\beta_2\sqrt{\alpha_1\gamma_1}\right),$$

and similarly

$$(15)\qquad \sqrt{s_1-s . s_2-s} = \tfrac{1}{2}\left(\beta_1\sqrt{\alpha_2\gamma_2}-\beta_2\sqrt{\alpha_1\gamma_1}\right).$$

With one condition still at disposal we can make

$$(16)\qquad \sqrt{\alpha_1\gamma_1} = \sqrt{\alpha_2\gamma_2} = \sqrt{s-\sigma},$$

$$(17)\qquad \frac{\beta_1}{\beta_2} = \frac{\sqrt{s_1-\sigma . s_2-\sigma}\pm\sqrt{s_1-s . s_2-s}}{\sqrt{s-\sigma}},$$

$$(18)\quad \frac{T_1}{T_2} = \frac{\pm t^4(L+L')\left(\sqrt{s_3-\sigma}\mp\sqrt{s_3-s}\right)\pm 2t^2\left(\sqrt{s_1-\sigma . s_2-\sigma}\pm\sqrt{s_1-s . s_2-s}\right)\pm(L-L')\sqrt{s_3-\sigma}\pm\sqrt{s_3-s}}{\sqrt{s-\sigma}}$$

suitable for $L^2 - L'^2 = s - \sigma$ positive; and then putting

$$(19)\qquad t^2\sqrt{\frac{L+L'}{L-L'}} = x,$$

$$(20)\qquad \frac{T}{(L-L')^2} = \left[x^2\frac{\sqrt{s_3-\sigma}-\sqrt{s_3-s}}{\sqrt{s-\sigma}} + 2x\frac{\sqrt{s_1-\sigma\,.\,s_2-\sigma}+\sqrt{s_1-s\,.\,s_2-s}}{s-\sigma} + \frac{\sqrt{s_3-\sigma}+\sqrt{s_3-s}}{\sqrt{s-\sigma}}\right]$$

$$\left[-x^2\frac{\sqrt{s_3-\sigma}+\sqrt{s_3-s}}{\sqrt{s-\sigma}} - 2x\frac{\sqrt{s_1-\sigma\,.\,s_2-\sigma}-\sqrt{s_1-s\,.\,s_2-s}}{s-\sigma} - \frac{\sqrt{s_3-\sigma}-\sqrt{s_3-s}}{\sqrt{s-\sigma}}\right]$$

homogeneous in σ and s, and so independent of homogeneity factor.

An appropriate change of sign will serve for $L^2 - L'^2 = s - \sigma$ negative. The same form of T_1 and T_1 will hold for parameters

$$(21)\qquad v, v' = \omega_1 + \frac{2r}{2n+1}\omega_3,$$

with an interchange of s_1 and s_3.

But with

$$(22)\qquad v, v' = \omega_1 + \frac{2r+1}{2n+1}\omega_3,$$

s_2 and s_3 change place, and $\sqrt{s_2-\sigma}, \sqrt{s_2-s}$ are negative, so that we take

$$(23)\qquad \begin{matrix}\alpha_1\\ \alpha_2\end{matrix} = (L+L')\frac{\sqrt{s-s_2}\mp\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}, \qquad (24)\qquad \begin{matrix}\gamma_1\\ \gamma_2\end{matrix} = -(L-L')\frac{\sqrt{s-s_2}\pm\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}},$$

$$(25)\qquad \begin{matrix}\beta_1\\ \beta_2\end{matrix} = -\frac{\sqrt{s_1-\sigma\,.\,\sigma-s_3}\pm\sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}}.$$

Finally, with

$$(26)\qquad v, v' = \omega_1 + \frac{2r+1}{2n}\omega_3,$$

s_2 and s_3 change place again in (23), (24), (25).

The discriminant of T_1 in (3), (18) as a quadratic in t^2

$$(27)\qquad \beta_1^2 - \alpha_1\gamma_1 = \frac{(\sqrt{s_1-\sigma\,.\,s_2-s}+\sqrt{s_1-s\,.\,s_2-\sigma})^2}{s-\sigma},$$

so that the roots can be expressed rationally by $\sqrt{s_a-\sigma}, \sqrt{s_a-s}$.

When the shape of the body is such that $p = p'$, the quartic Z reduces to a cubic in z, and our equations connecting $z = \cos\vartheta$ and ψ become the same as those required for the motion of a symmetrical top (*Annals of Mathematics*, vol. 5, 1904); in this case t^2+1 is a factor of T, say of T_1, so that

(28) $$\alpha_1 - 2\beta_1 + \gamma_1 = 0,$$

reducing in the case of $v = \omega_1 + f\omega_3$ to

(29) $$\begin{cases} (L+L')(\sqrt{s-s_2} - \sqrt{\sigma - s_2}) - (L-L')(\sqrt{s-s_2} + \sqrt{\sigma - s_2}) \\ \qquad = 2(\sqrt{s_1 - \sigma . \sigma - s_3} + \sqrt{s_1 - s . s - s_3}), \end{cases}$$

and then

(30) $$\begin{cases} (L+L')(\sqrt{s-s_2} - \sqrt{\sigma - s_2}) + (L-L')(\sqrt{s-s_2} + \sqrt{\sigma - s_2}) \\ \qquad = 2(\sqrt{s_1 - \sigma . s - s_3} + \sqrt{s_1 - s . \sigma - s_3}); \end{cases}$$

whence L and L' in a case where the motion is the same as top motion.

We can put

(31) $$L + L' = \sqrt{s-\sigma} . a, \quad L - L' = \sqrt{s-\sigma} . \frac{1}{a},$$

and then from (29), (30),

(32) $$a = \frac{(\sqrt{s_1-\sigma} + \sqrt{s_1 - s})(\sqrt{s-s_3} + \sqrt{\sigma - s_3})}{\sqrt{s-\sigma}(\sqrt{s-s_2} - \sqrt{\sigma - s_2})},$$

while

(33) $$M^2 = s\tfrac{1}{2}(v+v') - s\tfrac{1}{2}(v-v'),$$

denoting σ and s by $s(v)$ and $s(v')$.

18. Before proceeding further, test by a simple case

(1) $$v = \tfrac{1}{2}\omega_3,$$

so that we can take (*Phil. Trans.*, §37),

(2) $$s_1 = \frac{1}{\varkappa}, \qquad s_2 = \varkappa, \quad s_3 = 0, \quad \sigma = -1,$$

(3) $$P = \tfrac{1}{2}\left(\frac{1}{o} + o\right) = \frac{\mathrm{zs}\,\tfrac{1}{2}K'}{o}, \quad \text{with } \varkappa = o^2,$$

and now put

(4) $$s_3 - s = b^2, \qquad s_1 - s = \frac{1}{o^2} + b^2, \qquad s_2 - s = o^2 + b^2,$$

(5) $$B^2 = s_1 - s . s_2 - s = 1 + (4P^2 - 2)b^2 + b^4 = (1-b^2)^2 + 4P^2b^2,$$

(6) $$4P^2 = s_1 - \sigma . s_2 - \sigma,$$

(7) $$L^2 - L'^2 = s - \sigma = 1 - b^2.$$

Try

$$(8)\qquad \psi' = \tfrac{1}{2}\cos^{-1}\frac{t^2+K_1}{Nt^2},\quad \sqrt{T_1} = \tfrac{1}{2}\sin^{-1}\frac{t^2+K_2}{Nt^2}\sqrt{T_2},$$

with

$$(9)\qquad \alpha_1 = \alpha_2 = (L+L')^2,\qquad (10)\quad \gamma_1 = (1+b)^2,\quad \gamma_2 = (1-b)^2,$$

$$(11)\qquad \beta_1 = \frac{(2P+B)(1+b)}{L-L'},\qquad (12)\quad \beta_2 = \frac{(2P-B)(1-b)}{L-L'},$$

and then to satisfy the differentiation

$$(13)\qquad \frac{d\psi}{dt} = -\frac{(L+L')^2t^4 + 2(L+L')Pt^2 + L^2 - L'^2}{t\sqrt{T_1T_2}},$$

we find

$$(14)\qquad K_1 = -\frac{2Pb+B}{(L+L')(1+b)},\quad K_2 = \frac{2Pb+B}{(L+L')(1-b)}.$$

The results can be thrown into a more symmetrical shape by putting

$$(15)\qquad t^2 = x\sqrt{\frac{L-L'}{L+L'}}\left(\text{or } x\sqrt{\frac{L'-L}{L'+L}} \text{ if requisite}\right),$$

and writing

$$(16)\qquad \left\{\begin{aligned} \psi' &= \tfrac{1}{2}\cos^{-1}\frac{x\sqrt{\dfrac{-2Pb+B}{1-b}} - \sqrt{\dfrac{2Pb+B}{1+b}}}{Nx}\sqrt{X_1} \\ &= \tfrac{1}{2}\sin^{-1}\frac{x\sqrt{\dfrac{-2Pb+B}{1+b}} + \sqrt{\dfrac{2Pb+B}{1-b}}}{Nx}\sqrt{X_2}, \end{aligned}\right.$$

$$(17)\qquad X_1 = \quad x^2\sqrt{\frac{1-b}{1+b}} + 2x\frac{2P+B}{1-b^2} + \sqrt{\frac{1+b}{1-b}},$$

$$(18)\qquad X_2 = -x^2\sqrt{\frac{1+b}{1-b}} - 2x\frac{2P-B}{1-b^2} - \sqrt{\frac{1-b}{1+b}},$$

leading to

$$(19)\qquad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{(1-b^2)}} + 1}{2x\sqrt{X_1X_2}},$$

and, squaring and adding, we find

$$(20)\qquad N^2 = -\frac{16P}{(1-b^2)^{\frac{1}{2}}},$$

implying that X_1 and X_2 should change sign to obtain a real result.

If $L^2 - L'^2$, $s - \sigma$, and $1 - b^2$ are negative, so that it is requisite to take

(21) $$t^2 = x\sqrt{\frac{L' - L}{L' + L}},$$

then we write

(22) $$\left\{\begin{aligned} \psi &= \tfrac{1}{2}\cos^{-1}\frac{x\sqrt{\dfrac{B - 2Pb}{b - 1}} - \sqrt{\dfrac{B + 2Pb}{b + 1}}}{Nx}\sqrt{X_1} \\ &= \tfrac{1}{2}\sin^{-1}\frac{x\sqrt{\dfrac{B - 2Pb}{b + 1}} - \sqrt{\dfrac{B + 2Pb}{b - 1}}}{Nx}\sqrt{X_2}, \end{aligned}\right.$$

(23) $$X_1 = \quad x^2\sqrt{\frac{b - 1}{b + 1}} - 2x\frac{B + 2P}{b^2 - 1} + \sqrt{\frac{b + 1}{b - 1}},$$

(24) $$X_2 = -x^2\sqrt{\frac{b + 1}{b - 1}} + 2x\frac{B - 2P}{b^2 - 1} - \sqrt{\frac{b - 1}{b + 1}},$$

leading to

(25) $$\frac{d\psi}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{(b^2 - 1)}} - 1}{2x\sqrt{X_1X_2}},$$

and

(26) $$N^2 = -\frac{16P}{(b^2 - 1)^{\frac{3}{2}}}.$$

With a parameter

(27) $$v = \omega_1 + \tfrac{1}{2}\omega_3,$$

$s_3 - \sigma$ and $s_3 - s$ are negative, and we take

(28) $$\sigma = 1, \quad \sigma - s_3 = 1, \quad s - s_3 = b^2,$$

(29) $$L^2 - L'^2 = s - \sigma = b^2 - 1,$$

(30) $$B^2 = s_1 - s\,.\,s - s_2 = \left(\frac{1}{c^2} - b^2\right)(b^2 - c^2) = 4P^2b^2 - (b^2 - 1)^2,$$

(31) $$4P^2 = s_1 - \sigma\,.\,\sigma - s_2,$$

(32) $$P = \tfrac{1}{2}\left(\frac{1}{c} - c\right) = \frac{\operatorname{zn}\tfrac{1}{2}K'}{c}, \quad \kappa = c^2.$$

suppose; and this can be normalized to

(10) $$B = \left(\frac{1}{o^2} + b^2\right)(o^2 + b^2),$$

where o denotes the octahedron irrationality, defined by

(11) $$o^4 = x^2 = \frac{s_2 - s_3}{s_1 - s_3};$$

and with

(12) $$C = \left(\frac{1}{o^2} + c^2\right)(o^2 + c^2),$$

(13) $$X_1 = \quad x^2\sqrt{\frac{b-c}{b+c}} - 2x\frac{\sqrt{B} + \sqrt{C}}{b^2 - c^2} + \sqrt{\frac{b+c}{b-c}},$$

(14) $$X_2 = -x^2\sqrt{\frac{b+c}{b-c}} + 2x\frac{\sqrt{B} - \sqrt{C}}{b^2 - c^2} - \sqrt{\frac{b-c}{b+c}}.$$

But with

(15) $$L^2 - L'^2 = s - \sigma \text{ positive},$$

we put

(16) $$t^2 = x\sqrt{\frac{L - L'}{L + L'}};$$

b and c must change place, and

(17) $$\frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{s - \sigma}} + 1}{2x\sqrt{X_1 X_2}}.$$

With a parameter

(18) $$v = \omega_1 + \frac{\omega_3}{2n},$$

we have to put

(19) $$s - s_3 = b^2, \qquad \sigma - s_3 = c^2,$$

(20) $$B = s_1 - s \,.\, s - s_2 = (s_1 - s_3 - b^2)(b^2 - s_2 + s_3),$$

or normalized

(21) $$B = \left(\frac{1}{o^2} - b^2\right)(b^2 - o^2),$$

(22) $$C = \left(\frac{1}{o^2} - c^2\right)(c^2 - o^2),$$

and then

(23) $$X_1 = \quad x^2\sqrt{\frac{b-c}{b+c}} + 2x\frac{\sqrt{B} - \sqrt{C}}{b^2 - c^2} - \sqrt{\frac{b+c}{b-c}},$$

(24) $$X_2 = -x^2\sqrt{\frac{b+c}{b-c}} - 2x\frac{\sqrt{B} + \sqrt{C}}{b^2 - c^2} + \sqrt{\frac{b-c}{b+c}},$$

or else b and c, B and C change place, according as we have to put

$$(25)\qquad t^2 = x\sqrt{\frac{L-L'}{L+L'}},\quad \text{or } x\sqrt{\frac{L'-L}{L'+L}}.$$

In the determination of the coefficients $A_1, \ldots L_1, A_2, \ldots, L_2$, we can take

$$(26)\qquad A_1 = \frac{1}{\lambda}\sqrt[4]{\frac{b+c}{b-c}},\quad L_1 = -\lambda\sqrt[4]{\frac{b-c}{b+c}},$$

$$(27)\qquad A_2 = \frac{1}{\lambda}\sqrt[4]{\frac{b-c}{b+c}},\quad L_2 = -\lambda\sqrt[4]{\frac{b+c}{b-c}};$$

$$(28)\qquad A_1L_1 = A_2L_2 = -1,$$

and the value of λ may be inferred by induction from the associated pseudo-elliptic integral discussed in the *Phil. Trans.* 1904

$$(29)\qquad \begin{cases} I(v) = \displaystyle\int \frac{P(b^2 \sim c^2)+Q}{b^2 \sim c^2}\frac{db}{\sqrt{B}} \\ \quad = \displaystyle\frac{1}{n}\log\frac{E_1\sqrt{\tfrac12 B_1'} + E_2\sqrt{\tfrac12 B_2'}}{(b^2 \sim c^2)^{\frac12 n}},\quad B = B_1'B_2'; \end{cases}$$

from which we may take

$$(30)\qquad \lambda = e^{nI(v)} = \frac{E_1\sqrt{\tfrac12 B_1'} + E_2\sqrt{\tfrac12 B_2'}}{(b^2 \sim c^2)^{\frac12 n}}.$$

The remaining coefficients are now determined readily from the differentiation in (17).

Tested by $\mu = 8$ (*Phil. Trans.*, §38),

$$(31)\qquad v = \omega_1 + \tfrac14\omega_3,$$

$$(32)\qquad I(v) = \tfrac12\,\mathrm{ch}^{-1}\frac{b-c^2}{b^2-c^2}\sqrt{\tfrac12 B_1'} = \tfrac12\,\mathrm{sh}^{-1}\frac{b+c^2}{b^2-c^2}\sqrt{\tfrac12 B_2'},$$

$$(33)\qquad \begin{matrix} B_1' \\ B_2' \end{matrix} = \pm(b^2+1) + \left(\frac{1}{o}+o\right)b,$$

$$(34)\qquad \tfrac12\left(\frac{1}{o}+o\right) = \frac{\mathrm{zs}\,\tfrac12 K'}{o} = \left(\frac{c^2+1}{2c}\right)^2,$$

$$(35)\qquad P = \frac{\mathrm{zn}\,\tfrac14 K'}{o} = \frac{(3c^2+1)(c^2-1)}{8c^2},$$

$$(36)\qquad Q = \tfrac12(c^4-1) = c\sqrt{C};$$

and working with

$$(37)\qquad t^2 = x\sqrt{\frac{L-L'}{L+L'}},\quad L^2 - L'^2 = s - \sigma \text{ positive},$$

then

$$(38)\qquad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{b^2-c^2}} + 1}{2x\sqrt{X_1X_2}},$$

is satisfied by

$$(39)\qquad \begin{cases} \psi' = \frac14 \cos^{-1}(A_1x^3 + B_1x^2 + K_1x + L_1)\sqrt{X_1} \div Nx^2 \\ \quad = \frac14 \sin^{-1}(A_2x^3 + B_2x^2 + K_2x + L_2)\sqrt{X_2} \div Nx^2, \end{cases}$$

with X_1 and X_2 as in (23) and (24),

$$(40)\qquad \lambda = e^{2I} = \frac{(b^2-c^2)\sqrt{\frac12 B_1'} + (b^2+c^2)\sqrt{\frac12 B_2'}}{b^2-c^2},$$

and in addition to the values of A_1, L_1, A_2, L_2, we find

$$(41)\qquad \frac{B_1}{A_1} = \left(\frac{\sqrt{B}+\sqrt{C}}{b-c} + 2\frac{\sqrt{B}-\sqrt{C}}{b+c} - 4P\right) \div \sqrt{b^2-c^2},$$

$$(42)\qquad \frac{B_2}{A_2} = \left(2\frac{\sqrt{B}+\sqrt{C}}{b-c} + \frac{\sqrt{B}-\sqrt{C}}{b+c} - 4P\right) \div \sqrt{b^2-c^2},$$

$$(43)\qquad \frac{K_1}{L_1} = \left(-\frac{\sqrt{B}+\sqrt{C}}{b+c} - 2\frac{\sqrt{B}-\sqrt{C}}{b-c} - 4P\right) \div \sqrt{b^2-c^2},$$

$$(44)\qquad \frac{K_2}{L_2} = \left(-2\frac{\sqrt{B}+\sqrt{C}}{b+c} - \frac{\sqrt{B}-\sqrt{C}}{b-c} - 4P\right) \div \sqrt{b^2-c^2}.$$

$$(45)\qquad N^2 = -4\frac{(c^2+1)(c^4-1)^2}{(b^2-c^2)^4 c},$$

Tested numerically with

$$(46)\qquad b = \frac{\sqrt5+1}{2},\quad c = \frac{\sqrt5-1}{2},$$

we find

$$(47)\qquad \begin{cases} \psi' = \frac14\cos^{-1}\left(\dfrac{x^3}{\sqrt[4]{5}\sqrt[8]{5}} + x^2\dfrac{\sqrt5+1}{\sqrt[8]{5}} + x\dfrac{\sqrt5+1}{2}\sqrt[8]{5} - \sqrt[4]{5}\sqrt[8]{5}\right) \\ \qquad\qquad \sqrt{\left(x^2\sqrt[4]{5} + x - \dfrac{1}{\sqrt[4]{5}}\right)} \div Nx^2 \\ \quad = \frac14\sin^{-1}\left(\dfrac{x^3}{\sqrt[8]{5}} + x^2\dfrac{3+\sqrt5}{\sqrt[4]{5}\sqrt[8]{5}} + x\dfrac{\sqrt5+1}{2\sqrt[8]{5}} - \sqrt[8]{5}\right) \\ \qquad\qquad \sqrt{\left(-\dfrac{x^2}{\sqrt[4]{5}} - x\sqrt5 + \sqrt[4]{5}\right)} \div Nx^2, \end{cases}$$

leads to the requisite differentiation (38) with

$$(48)\qquad \frac{P}{\sqrt{(b^2-c^2)}} = \frac{1-2\sqrt5}{8\sqrt[4]{5}},\quad N = \frac{3-\sqrt5}{\sqrt[4]{5}}.$$

In this case, and also generally when $bc = 1$,

$$(49) \qquad v' = \omega_1 - \tfrac{3}{4}\omega_3, \quad v_1 = \omega_1 - \tfrac{1}{4}\omega_3, \quad v_2 = \tfrac{1}{2}\omega_3,$$

so that $\alpha_1, \beta_1, \gamma_1, \delta_1$, can be expressed by algebraical functions.

With $\mu = 12$ (*Phil. Trans.*, §40), $v = \omega_1 + \tfrac{1}{6}\omega_3$,

$$(50) \quad c^2 = a + a^2 + a^3,$$

$$(51) \quad \frac{B_1'}{B_2'} = \pm b^2 + \tfrac{1}{2}(1+a)^3(1-a)b \mp a^3(1+a+a^2),$$

$$(52) \quad \sqrt{C} = \tfrac{1}{2}(1-a^4)(a+a^2+a^3),$$

$$(53) \quad P = \tfrac{1}{12}(1-a)(5+3a+3a^2+a^3),$$

$$(54) \quad \lambda = \frac{[b^2+(1-a)b-1-a-a^2]\sqrt{\tfrac{1}{2}B_1'} + [b^2-(1-a)b-1-a-a^2]\sqrt{\tfrac{1}{2}B_2'}}{(b^2-a-a^2-a^3)^{\frac{1}{2}}},$$

and thence the calculation can be completed.

Material for the construction of cases corresponding to parameter

$$(55) \qquad v = \omega_1 + \tfrac{1}{8}\omega_3, \quad \omega_1 + \tfrac{1}{10}\omega_3, \quad \ldots\ldots\ldots,$$

will be found in *Phil. Trans.* 1904.

20. With $\mu = 4n + 2$, the method of reduction given in §24, *Phil. Trans.*, p. 251, must be followed; choosing a parameter

$$(1) \qquad v = \omega_1 + \frac{2r+1}{2n+1}\omega_3,$$

and using the new variable

$$(2) \qquad x = t^2\sqrt{\frac{L'+L}{L'-L}},$$

implying that $L^2 - L'^2$ and $s - \sigma$ are negative, we can write

$$(3) \quad \begin{cases} \psi' = \dfrac{1}{2n+1}\cos^{-1}\dfrac{A_1x^{2n}+B_1x^{2n-1}+\ldots+K_1x+L_1}{Nx^{n+\frac{1}{2}}}\sqrt{X_1} \\[2ex] = \dfrac{1}{2n+1}\sin^{-1}\dfrac{A_2x^{2n}+B_2x^{2n-1}+\ldots+K_2x+L_2}{Nx^{n+\frac{1}{2}}}\sqrt{X_2}, \end{cases}$$

satisfying the differentiation

$$(4) \qquad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{(\sigma-s)}} - 1}{2x\sqrt{X_1X_2}},$$

and

(5) $$X_1 = \alpha_1 x^2 + 2\beta_1 x + \gamma_1, \quad X_2 = -\alpha_2 x^2 - 2\beta_2 x - \gamma_2,$$

(6) $$\alpha_1 = \frac{\sqrt{\sigma - s_3} - \sqrt{s - s_2}}{\sqrt{\sigma - s}} = -\gamma_2,$$

(7) $$\alpha_2 = \frac{\sqrt{\sigma - s_3} + \sqrt{s - s_2}}{\sqrt{\sigma - s}} = -\gamma_1,$$

(8) $$\alpha_1\alpha_2 = \gamma_1\gamma_2 = 1,$$

(9) $$\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = \frac{\sqrt{s_1 - \sigma \,.\, \sigma - s_3} \pm \sqrt{s_1 - s \,.\, s - s_3}}{\sigma - s}.$$

Then

(10) $$A_1 = \frac{1}{\lambda}\sqrt{\alpha_2}, \quad L_1 = +\lambda\sqrt{\alpha_1},$$

(11) $$A_2 = \frac{1}{\lambda}\sqrt{\alpha_1}, \quad L_2 = -\lambda\sqrt{\alpha_2};$$

and guided by induction we find

(12) $$\lambda^2, \frac{1}{\lambda^2} = \frac{E\sqrt{s_1 - s \,.\, s - s_3} \pm F\sqrt{s - s_2}}{(\sigma - s)^{n+\frac{1}{2}}},$$

derivable from the associated pseudo-elliptic integral

(13) $$\begin{cases} I(v) = \displaystyle\int_{s_2}^{s} \frac{P(\sigma - s) - Q}{\sigma - s} \frac{ds}{\sqrt{(-S)}} \\ \quad = \dfrac{1}{2n+1} \operatorname{ch}^{-1} \dfrac{E\sqrt{s_1 - s \,.\, s - s_3}}{(\sigma - s)^{n+\frac{1}{2}}} = \dfrac{1}{2n+1} \operatorname{sh}^{-1} \dfrac{F\sqrt{s - s_2}}{(\sigma - s)^{n+\frac{1}{2}}}, \end{cases}$$

so that

(14) $$\lambda, \frac{1}{\lambda} = e^{\pm(n+\frac{1}{2})I(v)}.$$

Performing the differentiation of the $\cos^{-1}$ and $\sin^{-1}$ in (3),

(15) $$\begin{cases} \dfrac{d\psi}{dx} = -\dfrac{x^2 + 2x\dfrac{P}{\sqrt{\sigma - s}} - 1}{2x\sqrt{X_1X_2}} \\ \quad = -\dfrac{N_1}{2(A_2x^{2n} + \ldots + L_2)x\sqrt{X_1X_2}} = \dfrac{N_2}{2(A_1x^{2n} + \ldots + L_1)x\sqrt{X_1X_2}}, \end{cases}$$

and

$$(16)\quad \begin{cases} N_1 = [(2n-1)A_1x^{2n} + (2n-3)B_1x^{2n-1} + \ldots\ldots - (2n-1)K_1x - (2n+1)L_1](\alpha_1x^2 + 2\beta_1x + \gamma_1) \\ \quad + (A_1x^{2n} + B_1x^{2n-1} + \ldots\ldots + K_1x + L_1)(2\alpha_1x^2 + 2\beta_1x + 0) \\ = (A_2x^{2n} + B_2x^{2n-1} + \ldots\ldots + K_2x + L_2)\left[(2n+1)x^2 + 2x\dfrac{(2n+1)P}{\sqrt{\sigma - s}} - 2n - 1\right], \end{cases}$$

$$(17)\quad \begin{cases} N_2 = [(2n-1)A_2x^{2n} + (2n-3)B_2x^{2n-1} + \ldots\ldots - (2n-1)K_2x - (2n+1)L_2](-\alpha_2x^2 - 2\beta_2x - \gamma_2) \\ \quad + (A_2x^{2n} + Bx^{2n-1} + \ldots\ldots + K_2x + L_2)(-2\alpha_2x^2 - 2\beta_2x - 0) \\ = (A_1x^{2n} + B_1x^{2n-1} + \ldots\ldots + K_1x + L_1)\left[-(2n+1)x^2 - 2x\dfrac{(2n+1)P}{\sqrt{\sigma - s}} + 2n + 1\right], \end{cases}$$

and more than sufficient equations are obtained by equating cofficients in (16) and (17) for the determination of the $A, B, \ldots, K, L$, and for their verification, when s, s_1, s_2, s_3, σ and P are assigned for a given integer n.

Thus we find the values of A_1, A_2 and L_1, L_2 given above; and further

$$(18)\qquad \frac{B_1}{A_1} = (n - \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n + \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{\sigma - s}},$$

$$(19)\qquad \frac{B_2}{A_2} = (n + \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n - \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{\sigma - s}},$$

$$(20)\qquad \frac{K_1}{L_1} = -(n + \tfrac{1}{2})\frac{\beta_1}{\alpha_2} - (n - \tfrac{1}{2})\frac{\beta_2}{\alpha_1} + \frac{(2n+1)P}{\sqrt{\sigma - s}},$$

$$(21)\qquad \frac{K_2}{L_2} = -(n - \tfrac{1}{2})\frac{\beta_1}{\alpha_2} - (n + \tfrac{1}{2})\frac{\beta_2}{\alpha_1} + \frac{(2n+1)P}{\sqrt{\sigma - s}};$$

and so on.

If in addition

$$(22)\qquad v' = \omega_2, \quad s = s_2,$$

$$(23)\qquad X_1 = x^2 + 2x\frac{\sqrt{s_1 - \sigma\,.\,\sigma - s_3} + \sqrt{s_1 - s_2\,.\,s_2 - s_3}}{\sigma - s_2} - 1,$$

$$(24)\qquad X_2 = -x^2 + 2x\frac{-\sqrt{s_1 - \sigma\,.\,\sigma - s_3} + \sqrt{s_1 - s_2\,.\,s_2 - s_3}}{\sigma - s_2} + 1$$

and we can take

$$(25)\qquad A_1 = A_2 = 1, \quad L_1 = -L_2 = 1.$$

Thus, for example, utilising subsequent results in §22, we have the following cases; for $\mu = 6$,

$$(26)\begin{cases} \psi = \frac{1}{3}\cos^{-1}(x^2 + 0 + \qquad 1)\sqrt{\left(x^2 + 2x\dfrac{a+1}{\sqrt{2a-1}} - 1\right)} \div Nx^{\frac{3}{2}}, \\ \quad = \frac{1}{3}\sin^{-1}\left(x^2 + \dfrac{2x}{\sqrt{2a-1}} - 1\right)\sqrt{\left(x^2 - 2x\dfrac{a-1}{\sqrt{2a-1}} + 1\right)} \div Nx^{\frac{3}{2}}, \end{cases}$$

$$(27)\quad N = \frac{4a}{(2a-1)^{\frac{3}{4}}},$$

$$(28)\quad \frac{d\psi}{dx} = -\frac{x^2 + 2x\dfrac{\sqrt{2a-1}}{3} - 1}{2x\sqrt{\left[-x^4 - \dfrac{4ax^3}{\sqrt{2a-1}} - 2x^2\dfrac{2a^2-2a+1}{2a-1} + \dfrac{4ax}{\sqrt{2a-1}} - 1\right]}},$$

and putting

$$(29)\qquad x - \frac{1}{x} = y,$$

$$(30)\begin{cases} \psi = \frac{1}{3}\cos^{-1}\sqrt{(y^2+1)}\sqrt{\left(y + \dfrac{a+1}{\sqrt{2a-1}}\right)} \div N' \\ \quad = \frac{1}{3}\sin^{-1}\left(y + \dfrac{1}{\sqrt{2a-1}}\right)\sqrt{\left(-y - \dfrac{a-1}{\sqrt{2a-1}}\right)} \div N'. \end{cases}$$

For $\mu = 10$,

$$(31)\begin{cases} \psi = \frac{1}{5}\cos^{-1}\left[x^4 + 4x^3\dfrac{c-1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} - 2x^2\dfrac{(c-1)(c-3)}{-c^2+4c+1} - 4x\dfrac{c-1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} + 1\right] \\ \qquad\qquad \sqrt{\left[x^2 + 2x\dfrac{c^3-c^2+3c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} + 1\right]} \div Nx^{\frac{5}{2}} \\ \quad = \frac{1}{5}\sin^{-1}\left[x^2 + 2x\dfrac{(c-1)(c^2+1)}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} - 1\right](x^2+1) \\ \qquad\qquad \sqrt{\left[-x^2 + 2x\dfrac{c^3-c^2-5c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} + 1\right]} \div Nx^{\frac{5}{2}}, \end{cases}$$

or, putting

$$x - \frac{1}{x} = 2y,$$

$$(32)\begin{cases} \psi = \frac{1}{5}\cos^{-1}\dfrac{1}{N'}\left[y^2 + 2y\dfrac{c-1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1} - \dfrac{c^2-4c+1}{-c^2+4c+1}\right] \\ \qquad\qquad \sqrt{\left[y + \dfrac{c^3-c^2+3c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1}\right]} \\ \quad = \frac{1}{5}\sin^{-1}\dfrac{1}{N'}\left[y + \dfrac{(c-1)(c^2+1)}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1}\right]\sqrt{(y^2+1)} \\ \qquad\qquad \sqrt{\left[-y + \dfrac{c^3-c^2-5c+1}{(c+1)\sqrt{c^2-1}\,.\,-c^2+4c+1}\right]}, \end{cases}$$

$$(33)\quad N'^2 = \frac{128c^4}{(c+1)^4(-c^2+4c+1)^2\sqrt{(c^2-1)(-c^2+4c+1)}},$$

leading on differentiation to

$$(34) \quad \frac{d\psi'}{dy} = \frac{y + \dfrac{\frac{1}{3}(c+3)(-c^2+4c+1)}{(c+1)\sqrt{(c^2-1)(-c^2+4c+1)}}}{2\sqrt{y^2+1}\sqrt{y + \dfrac{c^3-c^2+3c+1}{(c+1)\sqrt{(c^2-1)(-c^2+4c+1)}}}\sqrt{-y + \dfrac{c^3-c^2-5c+1}{(c+1)\sqrt{(c^2-1)(-c^2+4c+1)}}}}.$$

Utilizing the results in *Phil. Trans.*, §§31, 32, we could go on to the construction of the case corresponding to $\mu = 14$,

$$(35) \quad \begin{cases} \psi' = \frac{1}{7}\cos^{-1} C(v^2 + E_1 v + E_2)\sqrt{v^2+1}\sqrt{v+\beta_1} \\ \quad = \frac{1}{7}\sin^{-1} C(v^3 + F_1 v^2 + F_2 v + F_3)\sqrt{-v-\beta_2}. \end{cases}$$

and corresponding to $\mu = 18$,

$$(36) \quad \begin{cases} \psi' = \frac{1}{9}\cos^{-1} C(v^4 + E_1 v^3 + E_2 v^2 + E_3 v + E_4)\sqrt{v+\beta_1} \\ \quad = \frac{1}{9}\sin^{-1} C(v^3 + F_1 v^2 + F_2 v + F_3)\sqrt{v^2+1}\sqrt{-v-\beta_2}. \end{cases}$$

21. But if $L^2 - L'^2 = s - \sigma$ is positive we have to use the new variable

$$(1) \quad x = t^2\sqrt{\frac{L+L'}{L-L'}},$$

and make the expression for ψ' in (3), §20, satisfy the differentiation

$$(2) \quad \frac{d\psi'}{dx} = -\frac{x^2 + 2x\dfrac{P}{\sqrt{s-\sigma}} + 1}{2x\sqrt{X_1 X_2}},$$

and

$$(3) \quad X_1 = \alpha_1 x^2 + 2\beta_1 x + \gamma_1, \quad X_2 = -\alpha_2 x^2 - 2\beta_2 x - \gamma_2,$$

$$(4) \quad \alpha_1 = \frac{\sqrt{s-s_2} - \sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} = -\gamma_2,$$

$$(5) \quad \alpha_2 = \frac{\sqrt{s-s_2} + \sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} = -\gamma_1,$$

$$(6) \quad \begin{matrix}\beta_1 \\ \beta_2\end{matrix} = -\frac{\sqrt{s_1-s\,.\,s-s_3} \pm \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{s-\sigma},$$

$$(7) \quad A_1 = \frac{\sqrt{\alpha_2}}{\lambda}, \quad L_1 = (-1)^r \lambda\sqrt{\alpha_1},$$

$$(8) \quad A_2 = \frac{\sqrt{\alpha_1}}{\lambda}, \quad L_2 = (-1)^r \lambda\sqrt{\alpha_2},$$

but now

$$(9) \quad \lambda^2 = \frac{F\sqrt{s-s_2} + E\sqrt{s_1-s\,.\,s-s_3}}{(s-\sigma)^{n+\frac{1}{2}}} = e^{(2n+1)I(v)},$$

derived from

$$(10)\quad \begin{cases} I(v) = \displaystyle\int \frac{P(s-\sigma)+Q}{s-\sigma}\frac{ds}{\sqrt{(-S)}} \\ = \dfrac{1}{2n+1}\,\mathrm{ch}^{-1}\dfrac{F\sqrt{s-s_2}}{(s-\sigma)^{n+\frac{1}{2}}} = \dfrac{1}{2n+1}\,\mathrm{sh}^{-1}\dfrac{E\sqrt{s_1-s\,.\,s-s_3}}{(s-\sigma)^{n+\frac{1}{2}}}. \end{cases}$$

In this case the degree can be halved by the substitution (*Phil. Trans.*, §24)'

$$(11)\quad q^2 = \frac{s-s_2}{s-\sigma},\quad q^2-1 = \frac{\sigma-s_2}{s-\sigma},$$

$$(12)\quad s-\sigma = \frac{\sigma-s_2}{q^2-1},\quad s-s_2 = (\sigma-s_2)\frac{q^2}{q^2-1},$$

$$(13)\quad s_1-s = \frac{(s_1-\sigma)q^2-(s_1-s_2)}{q^2-1},$$

$$(14)\quad s-s_3 = \frac{(\sigma-s_3)q^2-(s_2-s_3)}{q^2-1},$$

$$(15)\quad (s_1-s)(s-s_3) = \frac{(s_1-s_2)(s_2-s_3)}{(q^2-1)^2}\,Q_1Q_2,$$

$$(16)\quad \begin{matrix}Q_1\\Q_2\end{matrix} = C_2q^2 \pm C_1q + 1,$$

$$(17)\quad C_2 = \sqrt{\frac{s_1-\sigma\,.\,\sigma-s_3}{s_1-s_2\,.\,s_1-s_3}} = \frac{1}{2\sqrt{\alpha(\alpha-m+1)}},$$

$$(18)\quad C_1 = \sqrt{\frac{s_1-\sigma}{s_1-s_2}} + \sqrt{\frac{\sigma-s_3}{s_2-s_3}},$$

$$(19)\quad \alpha_1 = -\gamma_2 = q-\sqrt{q^2-1} = \left(\sqrt{\frac{q+1}{2}}-\sqrt{\frac{q-1}{2}}\right)^2,$$

$$(20)\quad \alpha_2 = -\gamma_1 = q+\sqrt{q^2-1} = \left(\sqrt{\frac{q+1}{2}}+\sqrt{\frac{q-1}{2}}\right)^2,$$

$$(21)\quad \begin{matrix}\beta_1\\\beta_2\end{matrix} = -\frac{\sqrt{s_1-s_2\,.\,s_2-s_3}\sqrt{Q_1Q_2} \pm \sqrt{s_1-\sigma\,.\,\sigma-s_3}(q^2-1)}{\sigma-s_2},$$

$$(22)\quad \begin{cases} I(v) = \displaystyle\int\left(q^2-1+2P\frac{\alpha-m}{m}\right)\frac{C_2dq}{\sqrt{(q^2-1)Q_1Q_2}} \\ = \dfrac{1}{n+\frac{1}{2}}\,\mathrm{ch}^{-1}E_1\sqrt{\dfrac{q+1}{2}}\sqrt{Q_1} = \dfrac{1}{n+\frac{1}{2}}\,\mathrm{sh}^{-1}E_2\sqrt{\dfrac{q-1}{2}}\sqrt{Q_2} \\ = \dfrac{1}{n+\frac{1}{2}}\log\lambda, \end{cases}$$

$$(23)\quad \lambda = e^{(n+\frac{1}{2})I(v)} = E_1\sqrt{\frac{q+1}{2}}\sqrt{Q_1} + E_2\sqrt{\frac{q-1}{2}}\sqrt{Q_2}.$$

Introducing the appropriate change of sign we have now

(24) $$\frac{B_1}{A_1} = (n - \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n + \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

(25) $$\frac{B_2}{A_2} = (n + \tfrac{1}{2})\frac{\beta_1}{\alpha_1} + (n - \tfrac{1}{2})\frac{\beta_2}{\alpha_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

(26) $$\frac{K_1}{L_1} = (n - \tfrac{1}{2})\frac{\beta_1}{\gamma_1} + (n + \tfrac{1}{2})\frac{\beta_2}{\gamma_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

(27) $$\frac{K_2}{L_2} = (n + \tfrac{1}{2})\frac{\beta_1}{\gamma_1} + (n - \tfrac{1}{2})\frac{\beta_2}{\gamma_2} - \frac{(2n+1)P}{\sqrt{s - \sigma}},$$

and so on.

Carry s up to s_1, so that

(28) $$v' = \omega_1, \quad v = \omega_1 + \frac{\omega_3}{2n+1},$$

(29) $$\frac{\alpha_1}{\alpha_2} = \frac{\sqrt{s_1 - s_2} \mp \sqrt{\sigma - s_2}}{\sqrt{s_1 - \sigma}} = \frac{-\gamma_2}{-\gamma_1},$$

(30) $$\frac{\beta_1}{\beta_2} = \mp \sqrt{\frac{\sigma - s_3}{s_1 - \sigma}},$$

and we can write

(31) $$X_1 = \quad x^2(\sqrt{s_1 - s_2} - \sqrt{\sigma - s_2}) - 2x\sqrt{\sigma - s_3} - \sqrt{s_1 - s_2} - \sqrt{\sigma - s_3},$$

(32) $$X_2 = -x^2(\sqrt{s_1 - s_2} + \sqrt{\sigma - s_2}) - 2x\sqrt{\sigma - s_3} + \sqrt{s_1 - s_2} - \sqrt{\sigma - s_3}.$$

22. Tested by (*Phil. Trans.*, §29),

(1) $$\mu = 6, \quad v = \omega_1 + \tfrac{1}{3}\omega_3,$$

(2) $$\sigma = 2c, \quad s_2 = 1, \quad P = \tfrac{1}{3}(2c - 1),$$

(3) $$s_1 - s \,.\, s - s_3 = -s^3 + (c^2 + 2c)s - c^2,$$

(4) $$s_1 - \sigma \,.\, \sigma - s_3 = c^2(2c - 1),$$

(5) $$\frac{Q_1}{Q_2} = -cq^2 \pm cq + 1,$$

(6) $$\lambda, \frac{1}{\lambda} = \sqrt{\frac{q+1}{2}}\sqrt{Q_1} \mp \sqrt{\frac{q-1}{2}}\sqrt{Q_2},$$

(7) $$\frac{\beta_1}{\beta_2} = -\frac{\sqrt{Q_1}\sqrt{Q_2} \pm c(q^2 - 1)}{\sqrt{2c - 1}},$$

(8) $$\frac{B_1}{B_2} = -\frac{(q+1)\sqrt{Q_2} \mp (q-1)\sqrt{Q_1}}{\sqrt{2c - 1}},$$

in

$$(9)\qquad \begin{cases} \psi' = \frac{1}{3}\cos^{-1}(A_1x^2 + B_1x + L_1)\sqrt{X_1} \div Nx^{\frac{3}{2}} \\ \quad = \frac{1}{3}\sin^{-1}(A_2x^2 + B_2x + L_2)\sqrt{X_2} \div Nx^{\frac{3}{2}}, \end{cases}$$

$$(10)\qquad \frac{d\psi}{dx} = -\frac{x^2 + \frac{2}{3}x\sqrt{(2c-1)(q^2-1)} + 1}{2x\sqrt{X_1X_2}},$$

with A, L, α, γ as in (7), (8), (19), (20) §21, and

$$(11)\qquad N^2 = \frac{16c^3(q^2-1)^3}{(2c-1)^{\frac{3}{2}}}.$$

Tested by (*Phil. Trans.*, §30),

$$\mu = 10, \quad v = \omega_1 + \tfrac{1}{5}\omega_3,$$

$$(12)\qquad P = \tfrac{1}{5}(c+3)(-c^2+4c+1),$$

$$(13)\qquad \sigma - s_2 = (c+1)^2(c^2-1)(-c^2+4c+1),$$

$$(14)\qquad (s_1-\sigma)(\sigma-s_3) = 16c^2(c+1)^2(c^2-1)(-c^2+4c+1),$$

$$(15)\qquad (s_1-s_2)(s_2-s_3) = (c^2-1)^5(-c^2+4c+1),$$

$$(16)\qquad \begin{matrix} Q_1 \\ Q_2 \end{matrix} = \frac{4cq^2 \pm 4c^2q}{(c+1)(c-1)^2} + 1,$$

$$(17)\qquad \lambda, \frac{1}{\lambda} = \left(\frac{2q}{c+1} - 1\right)\sqrt{\frac{q+1}{2}}\sqrt{Q_1} \pm \left(\frac{2q}{c+1} + 1\right)\sqrt{\frac{q-1}{2}}\sqrt{Q_2},$$

$$(18)\qquad \begin{matrix} \beta_1 \\ \beta_2 \end{matrix} = -\frac{(c+1)(c-1)^2\sqrt{Q_1}\sqrt{Q_2} \pm 4c(q^2-1)}{(c+1)\sqrt{(c^2-1)(-c^2+4c+1)}}.$$

Putting

$$(19)\qquad (q-1)\,[2c(c+3)q^2 - (c^3+3c^2-c+1)q + (c+1)(c^2+1)] = H_1,$$

$$(20)\qquad (q+1)\,[2c(c+3)q^2 + (c^3+3c^2-c+1)q + (c+1)(c^2+1)] = H_2,$$

$$(21)\qquad \sqrt{(q^2-1)}\,[2c(c+3)q^2 - (c^3+c^2-5c-1)q + 2(c+1)] = L_1,$$

$$(22)\qquad \sqrt{(q^2-1)}\,[2c(c+3)q^2 + (c^3+c^2-5c-1)q + 2(c+1)] = L_2,$$

$$(23)\qquad (q+1)\,[4cq^2 - 2c(c-3)q - (c+1)(c-3)] = M_1,$$

$$(24)\qquad (q-1)\,[4cq^2 + 2c(c-3)q - (c+1)(c-3)] = M_2,$$

$$(25)\qquad \frac{c+1}{c-1}\sqrt{\{(c+1)^3(c-1)(-c^2+4c+1)\}} = D,$$

we find

(26) $$DB_1 = (-H_1 - L_1)\sqrt{Q_1} + (+H_2 + L_2)\sqrt{Q_2},$$

(27) $$DB_2 = (+H_1 - L_1)\sqrt{Q_1} + (+H_2 - L_2)\sqrt{Q_2},$$

(28) $$DK_1 = (+H_1 - L_1)\sqrt{Q_1} + (-H_2 + L_2)\sqrt{Q_2},$$

(29) $$DK_2 = (-H_1 - L_1)\sqrt{Q_1} + (-H_2 - L_2)\sqrt{Q_2},$$

(30) $$C_1 = (M_1\sqrt{Q_1} + M_2\sqrt{Q_2})\frac{c-1}{(c+1)(-c^2+4c+1)},$$

(31) $$C_2 = (M_1\sqrt{Q_1} - M_2\sqrt{Q_2})\frac{c-1}{(c+1)(-c^2+4c+1)}$$

(32) $$N^2 = 2048c^4\left(\frac{c-1}{c+1}\right)^2\frac{(q^2-1)^5}{[(c^2-1)(-c^2+4c+1)]^2}.$$

A change of c into $-1/c$ will give a parameter

$$v = \omega_1 + \tfrac{3}{8}\omega_3 \text{ in the region } \sqrt{5}+2 > c > 1,$$

with

(33) $$P = \tfrac{1}{8}(3c-1)(-c^2+4c+1),$$

(34) $$\sigma - s_2 = (c-1)^2(c^2-1)(-c^2+4c+1),$$

(35) $$s_1 - \sigma \,.\, \sigma - s_3 = 16c^4(c-1)^2(c^2-1)(-c^2+4c+1),$$

(36) $$\lambda, \frac{1}{\lambda} = \pm\left(\frac{2cq}{c-1} - 1\right)\sqrt{\frac{q+1}{2}}\sqrt{Q_1} + \left(\frac{2cq}{c-1} + 1\right)\sqrt{\frac{q-1}{2}}\sqrt{Q_2},$$

(37) $$\frac{Q_1}{Q_2} = \frac{4c^2q^2 \mp 4cq}{(c-1)(c+1)^2} - 1,$$

(38) $$\frac{\beta_1}{\beta_2} = -\frac{(c-1)(c+1)^2\sqrt{Q_1}\sqrt{Q_2} \mp 4c^2(q^2-1)}{(c-1)\sqrt{(c^2-1)(-c^2+4c+1)}}, \text{ \&c.}$$

We can construct the special case in which

(39) $$v = \omega_1 + \tfrac{1}{8}\omega_3, \quad v' = \omega_1 + \tfrac{3}{8}\omega_3,$$

by taking

(40) $$L'^2 - L^2 = \sigma - s = 4c(c^2-1)(-c^2+4c+1),$$

and now

(41) $$\frac{q^2}{q^2-1} = \left(\frac{c+1}{c-1}\right)^2, \quad q = \frac{c+1}{2\sqrt{c}},$$

(42) $$\frac{q+1}{2}Q_1 = \frac{q-1}{2}Q_2 = \frac{1}{4\sqrt{c}}\cdot\frac{c-1}{c+1},$$

(43) $$\lambda = \sqrt[4]{c}\sqrt{\frac{c+1}{c-1}},$$

(44) $$\alpha_1 = \frac{1}{\sqrt{c}} = -\gamma_2, \quad \alpha_2 = \sqrt{c} = -\gamma_1,$$

(45) $$\beta_1 = \frac{c^2+1}{\sqrt{(c^2-1)(-c^2+4c+1)}}, \quad \beta_2 = \frac{-c^2+2c+1}{\sqrt{(c^2-1)(-c^2+4c+1)}},$$

and we find

$$(46)\begin{cases} \psi' = \frac{1}{5}\cos^{-1}\Big[x^4\sqrt{\frac{c-1}{c+1}} + x^3\frac{(c-1)^2(2c+1)}{(c+1)\sqrt{c}\sqrt{-c^2+4c+1}} \\ \quad + x^2\frac{-(c-1)(c-2)}{\sqrt{c^2-1}} + x\frac{2\sqrt{c}(c-2)}{\sqrt{-c^2+4c+1}} + \sqrt{\frac{c+1}{c-1}}\Big]\sqrt{X_1} \div Nx^{\frac{5}{2}}, \\ \quad = \frac{1}{5}\sin^{-1}\Big[\frac{x^4}{\sqrt{c}}\sqrt{\frac{c-1}{c+1}} + x^3\frac{(c-1)(3c+1)}{(c+1)\sqrt{-c^2+4c+1}} \\ \quad + x^2\frac{\sqrt{c}(c-1)(3c^2-8c+1)}{(-c^2+4c+1)\sqrt{c^2-1}} - x\frac{c^2-4c+1}{\sqrt{-c^2+4c+1}} - \sqrt{c}\sqrt{\frac{c+1}{c-1}}\Big]\sqrt{X_2} \div Nx^{\frac{5}{2}}, \end{cases}$$

$$(47)\qquad N^2 = \frac{2(c+1)^6}{c(c^2-1)^{\frac{1}{2}}(-c^2+4c+1)^{\frac{5}{2}}},$$

satisfying the differentiation (15), §20; at the same time ϕ' is also of a similar form, so that $\alpha_1, \beta_1, \gamma_1, \delta_1$, are quasi-algebraical.

A change of x into xi will give the case in which

$$(48)\qquad L^2 - L'^2 = s - \sigma = 4c(c^2-1)(c^2-4c-1),$$

is positive, as in the region

$$-\sqrt{5} + 2 > c > -1,$$

where

$$(49)\qquad v = \omega_1 + \tfrac{4}{5}\omega_3, \quad v' = \omega_1 + \tfrac{2}{5}\omega_3;$$

and now we take

$$(50)\ X_1 = \frac{x^2}{\sqrt{-c}} - 2x\frac{c^2+0+1}{\sqrt{(-c^2+1)(c^2-4c-1)}} - \sqrt{-c},$$

$$(51)\ X_2 = -x^2\sqrt{-c} + 2x\frac{c^2-2c-1}{\sqrt{(-c^2+1)(c^2-4c-1)}} + \frac{1}{\sqrt{-c}},$$

$$(52)\begin{cases} \psi' = \frac{1}{5}\cos^{-1}\Big[x^4\sqrt{\frac{-c+1}{c+1}} + x^3\frac{(-c+1)^2(2c+1)}{(c+1)\sqrt{-c}\sqrt{c^2-4c-1}} \\ \quad + x^2(c-2)\sqrt{\frac{-c+1}{c+1}} + x\frac{2\sqrt{-c}(c-2)}{\sqrt{c^2-4c-1}} - \sqrt{\frac{c+1}{-c+1}}\Big]\sqrt{X_1} \div Nx^{\frac{5}{2}} \\ \quad = \frac{1}{5}\sin^{-1}\Big[\frac{x^4}{\sqrt{-c}}\sqrt{\frac{-c+1}{c+1}} + x^3\frac{(-c+1)(3c+1)}{(c+1)\sqrt{c^2-4c-1}} \\ \quad - x^2\sqrt{-c}\frac{3c^2-8c+1}{c^2-4c-1}\sqrt{\frac{-c+1}{c+1}} - x\frac{c^2-4c+1}{\sqrt{c^2-4c-1}} - \sqrt{-c}\sqrt{\frac{c+1}{-c+1}}\Big]\sqrt{X_2} \div Nx^{\frac{5}{2}}, \end{cases}$$

$$(53)\qquad N^2 = \frac{2(c+1)^6}{-c(-c^2+1)^{\frac{1}{2}}(c^2-4c-1)^{\frac{5}{2}}},$$

satisfying, as in (2) §21,

$$(54)\qquad \frac{d\psi}{dx} = -\frac{x^2 + 2x\dfrac{(c+3)(c^2-4c-1)}{5\sqrt{-c}\sqrt{(-c^2-1)(c^2-4c-1)}} + 1}{2x\sqrt{X_1X_2}}.$$

According to *Phil. Trans.*, §31,

$$(55)\qquad \mu = 14, \quad v = \omega_1 + \tfrac{1}{7}\omega_3,$$

we have

$$(56)\qquad \begin{cases} \lambda = e^{\frac{1}{2}I(v)} \\ \quad = (A_2q^2 + A_1q + 1)\sqrt{\dfrac{q+1}{2}}\sqrt{Q_1} + (A_2q^2 - A_1q + 1)\sqrt{\dfrac{q-1}{2}}\sqrt{Q_2}, \end{cases}$$

$$(57)\qquad \begin{matrix} Q_1 \\ Q_2 \end{matrix} = C_2q^2 \pm C_1q + 1,$$

and putting $N_7 = 0$ in *Phil. Trans.* (16), p. 248,

$$(58)\qquad a = \frac{m(1-m)(2-3m+\sqrt{M})}{2(1-2m)^2}, \quad M = 4m - 11m^2 + 8m^3;$$

thence

$$(59)\qquad C_2 = \frac{(1-2m)(-m+\sqrt{M})}{4m(1-m)^3},$$

$$(60)\qquad C_1 = -\frac{(1-2m)(3m-4m^2+\sqrt{M})}{4m(1-m)^3},$$

$$(61)\qquad A_2 = \frac{-2+3m+\sqrt{M}}{2m(1-m)^2},$$

$$(62)\qquad A_1 = \frac{m-2m^2-\sqrt{M}}{2m(1-m)^2},$$

$$(63)\qquad 7P(v) = \frac{3m+2m^2-\sqrt{M}}{2m};$$

and here is material enough to build up the corresponding case of

$$(64)\qquad \begin{cases} \psi = \frac{1}{7}\cos^{-1}(A_1x^6 + B_1x^5 + \dots + K_1x + L_1)\sqrt{X_1} \div Nx^{\frac{7}{2}} \\ \quad = \frac{1}{7}\sin^{-1}(A_2x^6 + B_2x^5 + \dots + K_2x + L_2)\sqrt{X_2} \div Nx^{\frac{7}{2}}; \end{cases}$$

but A_1, A_2 and C_1, C_2 are used here in a meaning different to that above.

Similarly, from *Phil. Trans.*, §32, or *Archiv der Mathematik und Physik*, III, Serie I, p. 73, for

$$(65)\qquad \mu = 18, \quad v = \omega_1 + \tfrac{1}{9}\omega_3,$$

the expression for λ can be written down in the form

$$(66)\begin{cases}\lambda = e^{iI} \\ \quad =(A_3q^3+A_2q^2+A_1q+1)\sqrt{\frac{q+1}{2}}\sqrt{Q_1}+(A_3q^3-A_2q^2+A_1q-1)\sqrt{\frac{q-1}{2}}\sqrt{Q_2},\end{cases}$$

$$(67)\quad \begin{matrix}Q_1\\Q_2\end{matrix} = C_2q^2 \pm C_1q + 1,$$

and the values of A_1, A_2, A_3, and C_1, C_2, are given in *Phil. Trans.*, p. 271; and thence the construction of the case

$$(68)\qquad \begin{cases}\psi' = \frac{1}{3}\cos^{-1}(A_1x^8 + B_1x^7 + \ldots\ldots + K_1x + L_1)\sqrt{X_1} \div Nx^{\frac{9}{2}} \\ \quad = \frac{1}{3}\sin^{-1}(A_2x^8 + B_2x^7 + \ldots\ldots + K_2x + L_2)\sqrt{X_2} \div Nx^{\frac{9}{2}},\end{cases}$$

in which the two meanings of A_1, A_2, and C_1, C_2, must be kept distinct.

So also theoretically for $\mu = 22$, ...

23. For the determination of the motion of the centre O with μ even, we proceed as before for odd μ, and the expression for ψ' in terms of $t = \tan\frac{1}{2}\vartheta$ will serve for ϖ', if L' is taken to mean L'', when $k = \lambda^2$ is positive.

Replacing this t by $\dfrac{w+1}{-w+1}$, and putting

$$(1)\qquad \begin{cases}W_1 = (w-1)^4T_1 \\ \quad = \alpha_1(w+1)^4 + 2\beta_1(w^2-1)^2 + \gamma_1(w-1)^4 \\ \quad = (\alpha_1 + 2\beta_1 + \gamma_1)(w^4+1) \\ \quad + 4(\alpha_1 - \gamma_1)(w^3 + w) \\ \quad + (6\alpha_1 - 4\beta_1 + 6\gamma_1)w^2,\end{cases}$$

in which we can take, as a typical case,

$$(2)\qquad \alpha_1 = \quad (L + L')\frac{\sqrt{s-s_2} - \sqrt{\sigma - s_2}}{\sqrt{s-\sigma}},$$

$$(3)\qquad \gamma_1 = -(L - L')\frac{\sqrt{s-s_2} + \sqrt{\sigma - s_2}}{\sqrt{s-\sigma}},$$

$$(4)\qquad \beta_1 = -\frac{\sqrt{s_1-\sigma\,.\,\sigma-s_3} + \sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}},$$

so that

$$(5)\qquad \alpha_1 - \gamma_1 = 2\frac{L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}},$$

$$(6)\qquad \alpha_1 + \gamma_1 = 2\frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}},$$

$$(7)\quad \alpha_1 + 2\beta_1 + \gamma_1 = 2\frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-\sigma\,.\,\sigma-s_3} - \sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}},$$

$$(8)\ 6\alpha_1 - 4\beta_1 + 6\gamma_1 = 4\frac{3(L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2}) + \sqrt{s_1-\sigma\,.\,\sigma-s_3} + \sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}},$$

and so also for W_2; and then

$$(9)\qquad \frac{d\varpi'}{dw} = -2\frac{(L+P)(-w^2+1)^2+4L'(w^3+w)+8Lw^2}{(-w^2+1)\sqrt{(W_1W_2)}}.$$

But for negative k we replace w, L', and $\sqrt{s-s_2}$ by wi, $-L'i$, and $\sqrt{s_2-s}$; and obtain a real form for ϖ', leading to the differential relation

$$(10)\qquad \frac{d\varpi'}{dw} = -2\frac{(L+P)(w^2+1)^2-2L'(w^3-w)-8Lw^2}{(w^2+1)\sqrt{(W_1W_2)}},$$

24. Returning to the original variable z, and considering the resolution of the quartic

$$(1)\qquad \left\{\begin{aligned} Z &= az^4+4bz^3+6cz^2+4dz+e \\ &= a(z^2-1)\left(z^2-4\frac{B}{M}z-1-aD\right)-4\left(\frac{L'z-L}{M}\right)^2 \\ &= a(z^2-1)\left(z^2-4\frac{B}{M}z-1-aE\right)-4\left(\frac{Lz-L'}{M}\right)^2, \end{aligned}\right.$$

denote the roots by δ, α, β, γ, so that

$$(2)\qquad \delta+\alpha+\beta+\gamma = 4\frac{B}{M},$$

$$(3)\qquad (\delta+\alpha)(\beta+\gamma)+\delta\alpha+\beta\gamma = -2-aD-4a\frac{L'^2}{M^2} = -2-aE-4a\frac{L^2}{M^2},$$

$$(4)\qquad \delta(\beta\gamma+\gamma\alpha+\alpha\beta)+\alpha\beta\gamma = -4\frac{B}{M}-8a\frac{LL'}{M^2},$$

$$(5)\qquad \delta\alpha\beta\gamma = 1+aD-4a\frac{L^2}{M^2} = 1+aE-4a\frac{L'^2}{M^2}.$$

Then if x, y, denote any two values of z corresponding to the values u_1, u_2 of the elliptic argument

$$(6)\qquad u = \int\frac{dz}{\sqrt{Z}}.$$

Laguerre's formula gives

$$(7)\ \wp(u_1\pm u_2)-e_\alpha = \left[\frac{\sqrt{a.x-\delta.x-\alpha.y-\beta.y-\gamma)}\mp\sqrt{(a.y-\delta.y-\alpha.x-\beta.x-\gamma)}}{2(x-y)}\right].$$

Thus v_1 and v_2 denoting the values of u corresponding to $z = +1$ and $z = -1$,

$$(8)\quad \begin{cases} \wp(v_1 \pm v_2) - e_a \\ \qquad = \frac{1}{8}a[1 - (\delta + \alpha)(\beta + \gamma) + \delta\alpha + \beta\gamma + \delta\alpha\beta\gamma] \mp \dfrac{L^2 - L'^2}{2M^2} \\ \qquad = \frac{1}{8}a\left[1 - 2(\delta + \alpha)(\beta + \gamma) - 2 - aE - 4a\dfrac{L^2}{M^2} + 1 + aE - 4a\dfrac{L'^2}{M^2}\right] \\ \qquad\qquad \mp \dfrac{L^2 - L'^2}{2M^2} \\ \qquad = -\frac{1}{4}a(\delta + \alpha)(\beta + \gamma) - \dfrac{L^2 \text{ or } L'^2}{M^2}; \end{cases}$$

and denoting $v_1 + v_2$ by v and $v_1 - v_2$ by v', as before,

$$(9)\qquad \wp v - e_a = -\tfrac{1}{4}a(\delta + \alpha)(\beta + \gamma) - \frac{L^2}{M^2},$$

$$(10)\qquad \wp v' - e_a = -\tfrac{1}{4}a(\delta + \alpha)(\beta + \gamma) - \frac{L'^2}{M^2}.$$

The factor M is at our disposal as a *homogeneity factor*, so put

$$(11)\qquad \wp v - e_a = \frac{\sigma - s_a}{M^2},$$

in accordance with the notation employed in *Phil. Trans.*, 1904; then

$$(12)\qquad \tfrac{1}{4}M^2(\delta + \alpha)(\beta + \gamma) = -aL^2 - a(\sigma - s_a),$$

and

$$(13)\qquad \tfrac{1}{16}M^2[(\delta + \alpha) + (\beta + \gamma)]^2 = B^2,$$

so that

$$(14)\qquad \tfrac{1}{16}M^2[(\delta + \alpha) - (\beta + \gamma)]^2 = B^2 + aL^2 + a(\sigma - s_a) = N_a^2 \text{ suppose}$$

$$(15)\qquad \tfrac{1}{4}M[(\delta + \alpha) - (\beta + \gamma)] = N_a,$$

$$(16)\qquad \tfrac{1}{2}M(\delta + \alpha) = B - N_a,$$

$$(17)\qquad \tfrac{1}{2}M(\beta + \gamma) = B + N_a,$$

$$(18)\qquad M\delta = B - N_a - N_\beta - N_\gamma,$$

$$(19)\qquad M\alpha = B - N_a + N_\beta + N_\gamma,$$

$$(20)\qquad M\beta = B + N_a - N_\beta + N_\gamma,$$

$$(21)\qquad M\gamma = B + N_a + N_\beta - N_\gamma.$$

Then

$$(22)\qquad \wp v - e_\alpha = -\frac{aB^2 + L^2}{M^2} + \frac{aN_\alpha^2}{M^2},$$

and since from (19), §7,

$$(23)\qquad \wp 2w - \wp v = \frac{aB^2 + L^2}{M^2},$$

therefore

$$(24)\qquad \wp 2w - e_\alpha = \frac{aN_\alpha^2}{M^2},$$

w denoting the value of u corresponding to $z = \infty$.

Thus $v = 2w$ if $aB^2 + L^2 = 0$, implying $a = -1$.

Denoting $z - \delta \,.\, z - \alpha$ by Z_1 and $z - \beta \,.\, z - \gamma$ by Z_2,

$$(25)\qquad Z_1 = \left(z - \frac{B}{M}\right)^2 - 2\frac{N_\alpha}{M}\left(z - \frac{B}{M}\right) + \frac{N_\alpha^2 - (N_\beta + N_\gamma)^2}{M^2},$$

$$(26)\qquad Z_2 = \left(z - \frac{B}{M}\right)^2 + 2\frac{N_\alpha}{M}\left(z - \frac{B}{M}\right) + \frac{N_\alpha^2 - (N_\beta - N_\gamma)^2}{M^2},$$

and

$$(27)\qquad \begin{cases} Z = aZ_1Z_2 \\ \quad = a\left[\left(z - \frac{B}{M}\right)^4 - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2}\left(z - \frac{B}{M}\right)^2 - 8\frac{N_1N_2N_3}{M^3}\left(z - \frac{B}{M}\right)\right. \\ \quad \left. + \frac{N_1^4 + N_2^4 + N_3^4 - 2N_2^2N_3^2 - 2N_3^2N_1^2 - 2N_1^2N_2^2}{M^4}\right], \end{cases}$$

and, therefore, equating the coefficients of the powers of z,

$$(28)\qquad 6a\frac{B^2}{M^2} - 2a\frac{N_1^2 + N_2^2 + N_3^2}{M^2} = 6c = -2a - E - 4\frac{L^2}{M^2},$$

so that

$$(29)\qquad 2a\frac{N_1^2 + N_2^2 + N_3^2}{M^2} = 2a + E + 4\frac{L^2}{M^2} + 6a\frac{B^2}{M^2} = 6\wp 2w,$$

a verification;

$$(30)\qquad -4a\frac{B^3}{M^3} + 4a\frac{N_1^2 + N_2^2 + N_3^2}{M^2}\frac{B}{M} - 8a\frac{N_1N_2N_3}{M^3} = 4d = 4a\frac{B}{M} + 8\frac{LL'}{M^2},$$

$$(31)\qquad a\frac{B^4}{M^4} - 2\frac{N_1^2 + N_2^2 + N_3^2}{M^2}\frac{B^2}{M^2} + 8\frac{N_1N_2N_3}{M^3}\frac{B}{M} + \frac{N_1^4 + \ldots\ldots - 2N_2^2N_3^2\ldots\ldots}{M^4}$$

$$= e = a + E - 4\frac{L'^2}{M^2}.$$

From (10), §7,

$$(32)\qquad 2aBLL'M = iLM^3\wp'v - \tfrac{1}{2}EL^3M^2 = aL\surd - \Sigma - \tfrac{1}{2}EL^3M^2,$$

so that from (28),

$$(33)\qquad 2aBLL'M = aL\surd - \Sigma - L^2[a(N_1^2 + N_2^2 + N_3^2) - aM^2 - 2L^2 - 3aB^2],$$

and from (30)

$$(34)\qquad 2aBLL'M = -B^2M^2 - B^4 + (N_1^2 + N_2^2 + N_3^2)B^2 - 2N_1N_2N_3B,$$

so that

$$(35)\qquad \begin{cases} (B^2 + aL^2)M^2 = (B^2 + aL^2)(N_1^2 + N_2^2 + N_3^2) - B^4 - 3aB^2L^2 - 2L^4 \\ \qquad\qquad\qquad - 2B_1N_1N_2N_3 - aL\surd - \Sigma, \end{cases}$$

$$(36)\qquad \begin{cases} M^2 = N_1^2 + N_2^2 + N_3^2 - B^2 - 2aL^2 \dfrac{-2BN_1N_2N_3 + aL\surd - \Sigma}{B^2 + aL^2} \\ \quad = N_1^2 + N_2^2 + N_3^2 - B^2 - 2aL^2 + \dfrac{2aBN_1N_2N_3 - L\surd - \Sigma}{aB^2 + L^2}; \end{cases}$$

and we should obtain the same value of M^2 by equating the coefficients of z^0 in (27).

This determination of the homogeneity factor M^2 is essential in the applications, when we build up a solution on arbitrary values assigned to B, L, and σ, s_1, s_2, s_3, as will be seen in the sequel.

The expression for M^2 appears rather complicated and inelegant, but not much more than the form required in the corresponding case of the motion of a top (*Annals of Mathematics*, vol. 5, 1904).

The expression for M^2 can also be thrown into the form

$$(37)\qquad M^2 = \frac{(BN_\alpha - N_\beta N_\gamma)^2 - (L\sqrt{-\sigma_\alpha} + \sqrt{\sigma_\beta\sigma_\gamma})^2}{B^2 + aL^2 = N_\alpha^2 - a\sigma_\alpha} = a[s(2w) - s(v)].$$

Putting

$$(38)\qquad \wp(2w) - \wp(v) = s(2w) - s(v) = s' - \sigma,$$

and choosing $s_\alpha = s_2$, we can now write

$$(39)\qquad aM^2 = \frac{(B\sqrt{s' - s_2} - \sqrt{s_1 - s' . s' - s_3})^2 + (L\sqrt{s_2 - \sigma} - \sqrt{s_1 - \sigma . \sigma - s_3})^2}{s' - \sigma}.$$

25. Going back again to the variable t, of §12, and working with the case of $s_a = s_2$, and

$$(1)\quad \begin{cases} T_1 = \quad (L+L')\dfrac{\sqrt{s-s_2}-\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}\,t^4 \\ \qquad\qquad -2\dfrac{\sqrt{s_1-s\,.\,s-s_3}+\sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}\,t^2 \\ \qquad -(L-L')\dfrac{\sqrt{s-s_2}+\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} \end{cases}$$

$$(2)\quad \begin{cases} T_2 = -(L+L')\dfrac{\sqrt{s-s_2}+\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}\,t^4 \\ \qquad\qquad +2\dfrac{\sqrt{s_1-s\,.\,s-s_3}-\sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}\,t^2 \\ \qquad +(L-L')\dfrac{\sqrt{s-s_2}-\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}}, \end{cases}$$

then put

$$(3)\qquad t^2 = \frac{1-z}{1+z},\quad \frac{t^2}{t^2+1} = \frac{1-z}{2},\quad \frac{1}{t^2+1} = \frac{1+z}{2},$$

$$(4)\qquad M^2 Z = aM^2 Z_1 Z_2 = \frac{4T_1T_2}{(t^2+1)^4},$$

and identify with

$$(5)\quad \begin{cases} M_1 Z_1 = \dfrac{2T_1}{(t^2+1)^2} \\ \qquad = (L+L')\dfrac{\sqrt{s-s_2}-\sqrt{\sigma-s_2}}{2\sqrt{s-\sigma}}(z-1)^2 \\ \qquad\qquad +\dfrac{\sqrt{s_1-s\,.\,s-s_3}+\sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}(z^2-1) \\ \qquad -(L-L')\dfrac{\sqrt{s-s_2}+\sqrt{\sigma-s_2}}{2\sqrt{s-\sigma}}(z+1)^2, \end{cases}$$

and

$$(6)\qquad Z_1 = \left(z-\frac{B}{M}\right)^2 - 2\frac{N_2}{M}\left(z-\frac{B}{M}\right) + \frac{N_2^2-(N_1+N_3)^2}{M^2},$$

so that, equating coefficients of z^2, $-2z$, and z^0,

(7) $$M_1 = \frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} + \sqrt{s_1-s\,.\,s-s_3} + \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}},$$

(8) $$\frac{B+N_2}{M} = \frac{L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{M_1\sqrt{s-\sigma}},$$

(9) $$\left(\frac{B+N_2}{M}\right)^2 - \left(\frac{N_1+N_2}{M}\right)^2 = \frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-s\,.\,s-s_3} - \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{M_1\sqrt{s-\sigma}}.$$

Similarly, with

(10) $$M_2Z_2 = \frac{2T_2}{(t^2+1)^2},$$

(11) $$Z_2 = \left(z-\frac{B}{M}\right)^2 + 2\frac{N_2}{M}\left(z-\frac{B}{M}\right) + \frac{N_2^2-(N_1-N_3)^2}{M^2},$$

(12) $$M_2 = \frac{-L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-s\,.\,s-s_3} + \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}},$$

(13) $$\frac{B-N_2}{M} = \frac{-L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{M_2\sqrt{s-\sigma}},$$

(14) $$\left(\frac{B-N_2}{M}\right)^2 - \left(\frac{N_1-N_2}{M}\right)^2 = \frac{-L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} + \sqrt{s_1-s\,.\,s-s_3} - \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{M_2\sqrt{s-\sigma}}.$$

Then

(15) $$M_1 + M_2 = 2\frac{-L\sqrt{\sigma-s_2} + \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}},$$

(16) $$M_1 - M_2 = 2\frac{L'\sqrt{s-s_2} + \sqrt{s_1-s\,.\,s-s_3}}{\sqrt{s-\sigma}},$$

(17) $$\left\{\begin{aligned} aM^2 = M_1M_2 &= \frac{(L\sqrt{\sigma-s_2} - \sqrt{s_1-\sigma\,.\,\sigma-s_3})^2 - (L'\sqrt{s-s_2} + \sqrt{s_1-s\,.\,s-s_3})^2}{s-\sigma} \\ &= -L^2 + 2(s-\sigma) + 3M^2\wp v - \frac{L'\sqrt{-S} + Lxy}{s-\sigma}, \end{aligned}\right.$$

analogous to the form in (36), §24.

Also

(18) $$\left\{\begin{aligned} aM(B+N_2) &= \frac{L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} \\ &\times \frac{-L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} - \sqrt{s_1-s\,.\,s-s_3} + \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}, \end{aligned}\right.$$

(19) $$\left\{\begin{aligned} aM(B-N_2) &= \frac{-L\sqrt{s-s_2} - L'\sqrt{\sigma-s_2}}{\sqrt{s-\sigma}} \\ &\times \frac{L'\sqrt{s-s_2} - L\sqrt{\sigma-s_2} + \sqrt{s_1-s\,.\,s-s_3} + \sqrt{s_1-\sigma\,.\,\sigma-s_3}}{\sqrt{s-\sigma}}, \end{aligned}\right.$$

and adding,

(20) $$2aBM = -2LL' - \frac{L\sqrt{-S} + L'\sqrt{-\Sigma}}{s-\sigma}.$$

Subtracting

(21) $$\left\{\begin{aligned} 2MN_2 = {} & 2\sqrt{s-s_2 \,.\, \sigma - s_2} \\ & + 2\,\frac{-L\sqrt{s-s_2}\sqrt{s_1-\sigma \,.\, \sigma-s_3} - L'\sqrt{\sigma-s_2}\sqrt{s_1-s \,.\, s-s_3}}{s-\sigma}. \end{aligned}\right.$$

Again

(22) $$a(B+N_2)^2 - a(N_1+N_3)^2 = \frac{(L\sqrt{\sigma-s_2} + \sqrt{s_1-s \,.\, s-s_3})^2 - (L'\sqrt{s-s_2} - \sqrt{s_1-\sigma \,.\, \sigma-s_3})^2}{s-\sigma},$$

(23) $$a(B-N_2)^2 - a(N_1-N_3)^2 = \frac{(L\sqrt{\sigma-s_2} - \sqrt{s_1-s \,.\, s-s_3})^2 - (L'\sqrt{s-s_2} + \sqrt{s_1-\sigma \,.\, \sigma-s_3})^2}{s-\sigma},$$

and subtracting

(24) $$a(BN_2 - N_1N_3) = \frac{L\sqrt{\sigma-s_2}\sqrt{s_1-s \,.\, s-s_3} + L'\sqrt{s-s_2}\sqrt{s_1-\sigma \,.\, \sigma-s_3}}{s-\sigma},$$

while adding leads to an identity.

From the two equations (24) and

(25) $$2(aBM + LL') = -\frac{L\sqrt{-S} + L'\sqrt{-\Sigma}}{s-\sigma},$$

squaring and subtracting

(26) $$\left\{\begin{aligned} (aBM + LL')^2 - (BN_2 - N_1N_3)^2 &= \frac{L^2(s_1-s \,.\, s-s_3) - L'^2(s_1-\sigma \,.\, \sigma-s_3)}{s-\sigma}, \\ &= L^2(s_1 - s + \sigma - s_3) + (s_1-\sigma)(\sigma-s_3), \end{aligned}\right.$$

(27) $$\left\{\begin{aligned} 2aBLL'M = {} & -B^2M^2 - L^2L'^2 + (BN_2 - N_1N_3)^2 \\ & + L^2(s_1 - s + \sigma - s_3) + (s_1-\sigma)(\sigma-s_3) \end{aligned}\right.$$

and eliminating $\sqrt{-S}$ between (17) and (25)

(28) $$2aBLL'M = -aL^2M^2 - L^4 - L^2(3\sigma - s_1 - s_2 - s_3) - L\sqrt{-\Sigma},$$

so that, eliminating $2aBLL'M$,

(29) $$(B^2 + aL^2)M^2 = (BN_2 - N_1N_3)^2 + (L\sqrt{\sigma-s_2} - \sqrt{s_1-\sigma \,.\, \sigma-s_3})^2,$$

as before in (37), §24.

26. With $aB^2 + L^2 = 0$, implying $a = -1$, equation (20), §7, shows that

(1) $$\wp 2w - \wp v = 0, \quad 2w = v = v_1 + v_2,$$

and now, with

(2) $$\begin{cases} Z = -z^4 + 4bz^3 + 6cz^2 + 4dz + e \\ \quad = -(z^2 - 1)\left(z^2 - 4\frac{B}{M}z - 1 + E\right) - 4\left(\frac{Lz - L'}{M}\right)^2, \end{cases}$$

(3) $$b = \frac{B}{M}, \quad b + d = 2\frac{LL'}{M^2},$$

(4) $$1 + \frac{d}{b} = \pm \frac{2L'}{M}, \quad \text{as } B = \pm L,$$

(5) $$6c = 2 - E - 4b^2,$$

(6) $$e = -1 + E - \left(1 + \frac{d}{b}\right)^2,$$

(7) $$6c + e = 1 - 4b^2 - \left(1 + \frac{d}{b}\right)^2,$$

(8) $$\wp(v_1 - v_2) = -c - \tfrac{1}{4}\left(1 + \frac{d}{b}\right)^2,$$

(9) $$\wp(v_1 + v_2) = -c - b^2,$$

(10) $$i\wp'(v_1 - v_2) = \frac{(e + 1)(b + d) - 4b^2(b - d)}{\pm 4b}, \quad i\wp'(v_1 + v_2) = \frac{b^2 e + d^2}{\pm 2b},$$

whence

(11) $$\wp\tfrac{1}{2}(v_1 - v_2) = \tfrac{1}{2}\left(c + \frac{d}{b}\right),$$

(12) $$\wp\tfrac{1}{2}(v_1 + v_2) = \wp w = \tfrac{1}{2}(ad^2 + 2bd + c);$$

and

(13) $$\wp(u - w) = \frac{-z^2 + 2bz + c + \sqrt{(-Z)}}{2},$$

(14) $$\wp\left(u - \frac{v_1 + v_2}{2}\right) - \wp\frac{v_1 - v_2}{2} = \frac{-z^2 + 2bz - \frac{d}{b} + \sqrt{(-Z)}}{2}.$$

Now if we put

$$(15)\qquad \begin{cases} \chi = \tan^{-1}\dfrac{\sqrt{Z}}{-z^2+2bz-\dfrac{d}{b}} \\[2ex] \quad = \sin^{-1}\dfrac{\sqrt{Z}}{\sqrt{\left(\dfrac{d^2}{b^2}+e\right)}\sqrt{(-z^2+1)}} = \cos^{-1}\dfrac{-z^2+2bz-\dfrac{d}{b}}{\sqrt{\left(\dfrac{d^2}{b^2}+e\right)}\sqrt{(-z^2+1)}}, \end{cases}$$

we find on differentiation

$$(16)\qquad \frac{d\chi}{dz} = \frac{z}{\sqrt{Z}} + \frac{\left(1+\dfrac{d}{b}\right)z-2b}{(-z^2+1)\sqrt{Z}} = \frac{z}{\sqrt{Z}} \mp \frac{d(\psi-qFt)}{dz},$$

$$(17)\qquad \chi \pm (\psi - qFt) = \int\frac{zdz}{\sqrt{Z}},$$

according as $B = \pm L$; and

$$(18)\qquad i\left(z-\frac{B}{M}\right) = \zeta(v+w) - \zeta(v-w) - \zeta 2w;$$

so that

$$(19)\qquad \int\frac{zdz}{\sqrt{Z}} = \frac{B}{M}nt + i\log\frac{\sigma(v-w)}{\sigma(v+w)}e^{v\zeta 2w},$$

and algebraical cases can be constructed by choosing w an aliquot part of a period.

Now

$$(20)\qquad N_a^2 = s_a - \sigma = -\sigma_a \text{ suppose, and } 2N_1N_2N_3 = \sqrt{-\Sigma},$$

$$(21)\qquad \frac{2BN_1N_2N_3 - L\sqrt{-\Sigma}}{B^2-L^2} = \frac{0}{0} = \frac{2B^2\wp''v - \wp'^2v}{-2Bi\wp'v},$$

$$(22)\qquad M^2 = B^2 + N_1^2 + N_2^2 + N_3^2 + \frac{B^2\wp''v - \frac{1}{2}\wp'^2v}{Bi\wp'v},$$

$$(23)\qquad \begin{cases} BM^2 = B^3 + B^2\dfrac{\wp''v}{i\wp'v} - 3B\wp v + \frac{1}{2}i\wp'v \\[2ex] \quad = \left(B+\sqrt{-\dfrac{\sigma_2\sigma_3}{\sigma_1}}\right)\left(B+\sqrt{-\dfrac{\sigma_3\sigma_1}{\sigma_2}}\right)\left(B+\sqrt{-\dfrac{\sigma_1\sigma_2}{\sigma_3}}\right), \end{cases}$$

and from (34), §24,

$$(24)\qquad \left\{\begin{aligned} 2LL'M &= BM^2 + B^3 + (N_1^2 + N_2^2 + N_3^2)B - 2N_1N_2N_3 \\ &= 2B^3 - B^2\frac{\wp''v}{i\wp'v} + \tfrac{1}{2}i\wp'v. \end{aligned}\right.$$

With

$$(25)\qquad N_\alpha = 0, \qquad aB^2 + L^2 + \sigma - s_\alpha = 0,$$

$$(26)\qquad Z_1 = \left(z - \frac{B}{M}\right)^2 - \left(\frac{N_\beta + N_\gamma}{M}\right)^2, \quad Z_2 = \left(z - \frac{B}{M}\right)^2 - \left(\frac{N_\beta - N_\gamma}{M}\right)^2,$$

$$(27)\qquad \wp 2w - e_\alpha = 0, \qquad \wp' 2w = 0,$$

$$(28)\qquad 2b^3 - 3abc + a^2d = 0,$$

$$(29)\qquad \left\{\begin{aligned} -aM^2 &= L^2 + 2aL\sqrt{\frac{s_\beta - \sigma \cdot s_\gamma - \sigma}{s_\alpha - \sigma}} - s_\alpha + s_\beta + s_\gamma - \sigma \\ &= L^2 + 2aL\sqrt{\frac{\sigma_\beta\sigma_\gamma}{\sigma_\alpha}} - \sigma_\alpha + \sigma_\beta + \sigma_\gamma \\ &= \left(L + a\sqrt{\frac{\sigma_\beta\sigma_\gamma}{\sigma_\alpha}}\right)^2 - \frac{\sigma_\alpha - \sigma_\beta \cdot \sigma_\alpha - \sigma_\gamma}{\sigma_\alpha}, \end{aligned}\right.$$

27. With $\mu = 2n$, an even number, and a parameter

$$(1)\qquad v = \frac{\omega_3}{n}, \text{ or } \omega_1 + \frac{\omega_3}{n},$$

we can write

$$(2)\qquad \left\{\begin{aligned} \psi &= \frac{1}{n}\cos^{-1}\frac{Hz^{n-1} + H_1z^{n-2} + \dots}{(-z^2+1)^{\frac{1}{2}n}}\sqrt{Z_1(-1)^n} \\ &= \frac{1}{n}\sin^{-1}\frac{Kz^{n-1} + K_1z^{n-2} + \dots}{(-z^2+1)^{\frac{1}{2}n}}\sqrt{aZ_2(-1)^n}, \end{aligned}\right.$$

and determine the coefficients from the differential relation of (38), §9.

It is of great assistance in the calculation to start with a knowledge of the the leading coefficients H and K, and these can be determined readily by taking $z = \infty$, when

$$(3)\qquad e^{n\psi' i} = H + K\sqrt{-a},$$

$$(4)\qquad e^{2n\psi' i} = \frac{H + K\sqrt{-a}}{H - K\sqrt{-a}},$$

since

$$(5)\qquad H^2 + K^2a = 1.$$

But now at $z = \infty$; from (18), §9,

$$\psi i = \tfrac{1}{2} i I(2w, v) - \tfrac{1}{4} \log \frac{\wp(v_1 - w) - \wp(v_2 - w)}{\wp(v_1 + w) - \wp(v_2 + w)}, \tag{6}$$

and from (25), (26), §9,

$$\wp(v_1 - w) - \wp(v_2 - w) = -2\alpha \frac{B}{M} - 2 \frac{L}{M} \sqrt{-a}, \tag{7}$$

$$\wp(v_1 + w) - \wp(v_2 + w) = -2\alpha \frac{B}{M} + 2 \frac{L}{M} \sqrt{-a}, \tag{8}$$

so that

$$e^{2n\psi' i} = e^{nIi} \left(\frac{B\sqrt{-a} + L}{B\sqrt{-a} - L} \right)^{\frac{1}{2}n} \tag{9}$$

writing I for $I\,(2w, v)$.

With a parameter v as in (1), and

$$M^2\,(\wp 2w - \wp v) = s - \sigma = aB^2 + L^2, \tag{10}$$

we have I given by an expression of the form (*Phil. Trans.*, 1904.)

$$I = \frac{1}{n} \cos^{-1} \frac{G N_\beta N_\gamma}{(\sigma - s)^{\frac{1}{2}n}} = \frac{1}{n} \sin^{-1} \frac{F N_\alpha \sqrt{a}}{(\sigma - s)^{\frac{1}{2}n}}, \tag{11}$$

$$e^{nIi} = \frac{G N_\beta N_\gamma + F N_\alpha \sqrt{-a}}{(\sigma - s)^{\frac{1}{2}n}}, \tag{12}$$

$$e^{2nIi} = \frac{G N_\beta N_\gamma + F N_\alpha \sqrt{-a}}{G N_\beta N_\gamma - F N_\alpha \sqrt{-a}}, \tag{13}$$

so that

$$H + K\sqrt{-a} = \frac{\sqrt{(G N_\beta N_\gamma + F N_\alpha \sqrt{-a})}}{(B\sqrt{-a} + L)^{\frac{1}{2}n}}, \tag{14}$$

$$H - K\sqrt{-a} = \frac{\sqrt{(G N_\beta N_\gamma - F N_\alpha \sqrt{-a})}}{(B\sqrt{-a} - L)^{\frac{1}{2}n}}, \tag{15}$$

whence H and K, in the form

$$H = \frac{\cos}{\text{ch}} \frac{1}{2} n (I + \beta), \quad K = \frac{\sin}{\text{sh}} \frac{1}{2} n (I + \beta), \tag{16}$$

on putting

$$B = \sqrt{s - \sigma} \frac{\cos}{\text{ch}} \beta, \quad L = \sqrt{s - \sigma} \frac{\sin}{\text{sh}} \beta. \tag{17}$$

When

(18) $$aB^2+L^2=0,\ a=-1,\ B=L,\ s=\sigma,$$

(19) $$H+K=\frac{\sqrt{(2FN_\alpha)}}{(2L)^{\frac{1}{2}n}},\ H-K=\frac{(2L)^{\frac{1}{2}n}}{\sqrt{(2FN_\alpha)}},$$

(20) $$H,K=\frac{\pm(2L)^n+2FN_\alpha}{2(2L)^{\frac{1}{2}n}\sqrt{(2FN_\alpha)}}.$$

From the differential relation (37), §9, we find

(21) $$H_1=H\frac{B+N_\alpha}{M}+nK\frac{L-P}{M},\ K_1=K\frac{B-N_\alpha}{M}-nH\frac{L-P}{M},$$

(22) $$\begin{cases} H_2=H\left[\left(\frac{B+N_\alpha}{M}\right)^2+\frac{1}{2}\left(\frac{N_\beta-N_\gamma}{M}\right)^2-\frac{1}{2}n-\frac{1}{2}n^2\left(\frac{L-P}{M}\right)^2\right] \\ \quad +nK\left(-\frac{L'}{M}+\frac{2B+N_\alpha}{M}\frac{L-P}{M}\right), \\ K_2=K\left[\left(\frac{B-N_\alpha}{M}\right)^2+\frac{1}{2}\left(\frac{N_\beta+N_\gamma}{M}\right)^2-\frac{1}{2}n-\frac{1}{2}n^2\left(\frac{L-P}{M}\right)^2\right] \\ \quad +nH\left(-\frac{L'}{M}+\frac{2B-N_\alpha}{M}\frac{L-P}{M}\right), \end{cases}$$

and so on.

28. Test by $v=\frac{1}{2}\omega_3$, $a=-1$, as in §18, from which the result in terms of z can be obtained by putting

$$t^2=\frac{1-z}{1+z};$$

but independently, starting with z as the variable,

(1) $$\psi=\frac{1}{2}\cos^{-1}\frac{Hz+H_1}{-z^2+1}\sqrt{Z_1}=\frac{1}{2}\sin^{-1}\frac{Kz+K_1}{-z^2+1}\sqrt{-Z_2},$$

(2) $$Z_1=\left(z-\frac{B+N_3}{M}\right)^2-\left(\frac{N_1-N_2}{M}\right)^2$$

(3) $$Z_2=\left(z-\frac{B-N_3}{M}\right)^2-\left(\frac{N_1+N_2}{M}\right)^2,$$

(4) $$-B^2+L^2=s-\sigma=s+1,$$

so that

(5) $$N_3^2=B^2-L^2+1=-s=b^2,$$

suppose

(6) $$N_1^2 = b^2 + \frac{1}{o^2},\ N_2^2 = b^2 + o^2,$$

and with

(7) $$\frac{1}{o} + o = 2P,$$

(8) $$e^{2K} = \frac{N_1 N_2 + 2Pb}{b^2 - 1} = \frac{\sqrt{(N_1 N_2 + b^2 - 1)} + \sqrt{(N_1 N_2 - b^2 + 1)}}{\sqrt{(N_1 N_2 + b^2 - 1)} - \sqrt{(N_1 N_2 - b^2 + 1)}},$$

(9) $$H + K = \frac{\sqrt{(N_1 N_2 + b^2 - 1)} + \sqrt{(N_1 N_2 - b^2 + 1)}}{\sqrt{2(B - L)}},$$

(10) $$H - K = \frac{\sqrt{(N_1 N_2 + b^2 - 1)} - \sqrt{(N_1 N_2 - b^2 + 1)}}{\sqrt{2(B + L)}},$$

whence H and K; and

(11) $$M^2 = 2L^2 - B^2 + 3b^2 + 4P^2 - 2 + 2\frac{\sqrt{[(b^2-1)^2 + 4P^2b^2]} + 2PL}{B^2 - L^2}.$$

Then

(12) $$Hz + H_1 = H\left(z + \frac{B + N_3}{M}\right) - 2K\frac{L - P}{M},$$

(13) $$Kz + K_1 = K\left(z + \frac{B - N_3}{M}\right) - 2H\frac{L - P}{M},$$

and the verification can be completed.

With

(14) $$B^2 - L^2 = 0,\ B = L,\ b = 1,$$

(15) $$N_1^2 = 1 + \frac{1}{o^2},\ N_2^2 = 1 + o^2,\ N_3^2 = 1,$$

(16) $$LM^2 = \left(L - \frac{1}{o}\right)(L - o)\left(L - \frac{1}{o} - o\right) = (L^2 - 2PL + 1)(L - 2P),$$

(17) $$H, K = \frac{\pm L^2 + P}{L\sqrt{2P}}.$$

Try

(18) $$H = 1,\ K = 0,\ P = L^2,$$

(19) $$M^2 = (L - 1)(2L - 1)(2L^2 + L + 1),$$

with

(20) $$N_3 = 0,\ b = 0,\ s = s_3,\ L^2 - B^2 = 1,\ N_1 = \frac{1}{o},\ N_2 = o,$$

(21) $$H = L,\ K = B,\ M^2 = (L - 2P)^2 - 1.$$

In Kirchhoff's case of

(22) $B = 0,\quad L^2 = 1,\quad L = -1,\quad L' = 0,\quad M^2 = 4P(P+1),\quad H = 1,\quad K = 0,$

(23) $\psi' = \frac{1}{2}\cos^{-1}\frac{z\sqrt{(z^2 - 1 + \frac{1}{P})}}{-z^2+1} = \frac{1}{2}\sin^{-1}\frac{\sqrt{(-\frac{P+1}{P}z^2 + 1)}}{-z^2+1},$

satisfying

(24) $$\frac{d\psi'}{dz} = \frac{-(P+1)z^2 + P - 1}{M(-z^2+1)\sqrt{Z}}.$$

A change of sign of o^2 will change the parameter v from $\frac{1}{2}\omega_3$ to $\omega_1 + \frac{1}{2}\omega_3$, still with $a = -1$; and now

(25) $$N_1^2 = B^2 - L^2 + \frac{1}{o^2} - 1,\; N_2^2 = B^2 - L^2 - 1 + o^2,\; N_3^2 = B^2 - L^2 - 1,$$

(26) $\frac{1}{o} - o = 2P.$

As a special case, take

(27) $$N_3 = 0,\qquad B^2 - L^2 = 1.$$

Again, still further, take

(28) $$L = 0,\qquad B^2 = 1,\qquad H = 1,\qquad K = 0.$$

(29) $$\left\{\begin{aligned} \psi' &= \tfrac{1}{2}\cos^{-1}\frac{\left(z + \frac{B}{M}\right)\sqrt{\left[\left(z - \frac{B}{M}\right)^2 - 4\frac{P^2}{M^2}\right]}}{-z^2+1} \\ &= \tfrac{1}{2}\sin^{-1}\frac{\frac{2P}{M}\sqrt{\left[-\left(z - \frac{B}{M}\right)^2 + 4\frac{P^2+1}{M^2}\right]}}{-z^2+1}, \end{aligned}\right.$$

(30) $$M^2 = \frac{1}{o^2} - 1 + o^2 = 4P^2 + 1,\quad \frac{L'}{M} = -\frac{2BP}{4P^2+1}.$$

This case can be written, putting $\frac{B}{M} = b$,

(31) $$\begin{aligned}(-z^2 + 1)e^{2\psi' i} = (z+b)\quad &\sqrt{[(z-b)^2 - 1 + b^2]} \\ &+ i\sqrt{(1-b^2)}\sqrt{[-(z-b)^2 + 1 + 3b^2]}.\end{aligned}$$

29. Next, with

(1) $$v = \omega_1 + \tfrac{1}{2}\omega_3;\quad a = 1,$$

(2) $$N_3^2 = B^2 + L^2 + 1 = b^2,$$

suppose; so that

(3) $$N_1^2 = B^2 + L^2 - \frac{1}{o^2} + 1 = b^2 - \frac{1}{o^2},$$

(4) $$N_2^2 = B^2 + L^2 + 1 - o^2 = b^2 - o^2.$$

Now we find, with

(5) $$\frac{1}{o} - o = 2P, \sqrt{(b^2 - 1 + N_1 N_2)} = \sqrt{\frac{b^2 + 2Pb - 1}{2}} + \sqrt{\frac{b^2 - 2Pb - 1}{2}},$$

(6) $$H - Ki = \frac{(1+i)\sqrt{(b^2 + 2Pb - 1)} + (1-i)\sqrt{(b^2 - 2Pb - 1)}}{2(B - Li)},$$

(7) $$H + Ki = \frac{(1-i)\sqrt{(b^2 + 2Pb - 1)} + (1+i)\sqrt{(b^2 - 2Pb - 1)}}{2(B + Li)},$$

(8) $$H = \frac{(-B + L)\sqrt{(b^2 + 2Pb - 1)} - (B + L)\sqrt{(b^2 - 2Pb - 1)}}{2(b^2 - 1)},$$

(9) $$K = \frac{(B + L)\sqrt{(b^2 + 2Pb - 1)} - (B - L)\sqrt{(b^2 - 2Pb - 1)}}{2(b^2 - 1)},$$

(10) $$\psi = \tfrac{1}{2}\cos^{-1}\frac{Hz + H_1}{-z^2 + 1}\sqrt{Z_1} = \tfrac{1}{2}\sin^{-1}\frac{Kz + K_1}{-z^2 + 1}\sqrt{Z_2},$$

(11) $$\begin{cases} Z_1 = \left(z - \dfrac{B + N_3}{M}\right)^2 - \left(\dfrac{N_1 - N_2}{M}\right)^2, \\ Z_2 = \left(z - \dfrac{B - N_3}{M}\right)^2 - \left(\dfrac{N_1 + N_2}{M}\right)^2. \end{cases}$$

(12) $$\begin{cases} M^2 = -2L^2 - B^2 + N_1^2 + N_2^2 + N_3^2 + 2\dfrac{BN_1 N_2 N_3 - 2PL}{B^2 + L^2} \\ \quad = -L^2 + 2b^2 - 4P^2 - 1 + 2\dfrac{Bb\sqrt{[(b^2 - 1)^2 - 4P^2 b^2]} - 2PL}{b^2 - 1}. \end{cases}$$

From the differential relation (38), §9

(13) $$Hz + H_1 = H\left(z + \frac{B + N_3}{M}\right) - 2K\frac{L - P}{M},$$

(14) $$Kz + K_1 = K\left(z + \frac{B - N_3}{M}\right) + 2H\frac{L - P}{M}.$$

As a special case, try

(15) $$N_1 = 0, \quad B^2 + L^2 = \frac{1}{o^2} - 1, \quad N_3 = b = \frac{1}{o},$$

(16) $$H = \frac{(-B + L)o}{\sqrt{2}\sqrt{(1 - o^2)}}, \quad K = \frac{(B + L)o}{\sqrt{2}\sqrt{(1 - o^2)}},$$

(17) $$M^2 = \frac{1}{o^2} + 1 - (L + o)^2;$$

and then in Kirchhoff's case

$$(18)\qquad B=0,\quad H=K=\frac{1}{\sqrt{2}},\quad L=\sqrt{\left(\frac{1}{\sigma^2}-1\right)},\quad M=1-\sqrt{1-\sigma^2}.$$

$$(19)\quad\begin{cases}\psi'=\frac{1}{2}\cos^{-1}\dfrac{\left(z+\dfrac{1-\sqrt{1-\sigma^2}}{\sigma}\right)\sqrt{\left[z^2-2z\dfrac{1+\sqrt{1-\sigma^2}}{\sigma^3}+\left(\dfrac{1+\sqrt{1-\sigma^2}}{\sigma}\right)^2\right]}}{\sqrt{2}(-z^2+1)}\\[2ex] \quad=\frac{1}{2}\sin^{-1}\dfrac{\left(z-\dfrac{1-\sqrt{1-\sigma^2}}{\sigma}\right)\sqrt{\left[z^2+2z\dfrac{1+\sqrt{1-\sigma^2}}{\sigma^3}+\left(\dfrac{1+\sqrt{1-\sigma^2}}{\sigma}\right)^2\right]}}{\sqrt{2}(-z^2+1)}\end{cases}$$

leading on differentiation to

$$(20)\qquad \frac{d\psi'}{dz}=\frac{\dfrac{\sqrt{1-\sigma^2}}{2\sigma^3}\left[(1+\sigma^2+\sqrt{1-\sigma^2})z^2+3-\sigma^2+3\sqrt{1-\sigma^2}\right]}{(-z^2+1)\sqrt{Z}},$$

so that $L'=0$.

But if $L=0$ instead of B when $N_1=0$,

$$(21)\qquad -H=K=\frac{1}{\sqrt{2}},\quad B=\sqrt{\left(\frac{1}{\sigma^2}-1\right)},\quad M^2=\frac{1}{\sigma^2}+1-\sigma^2,$$

$$(22)\quad\begin{cases}\psi'=\frac{1}{2}\cos^{-1}\dfrac{\left(-z-\dfrac{\sigma^2+\sqrt{1-\sigma^2}}{\sqrt{1+\sigma^2-\sigma^4}}\right)\sqrt{\left[z^2-2z\dfrac{1+\sqrt{1-\sigma^2}}{\sqrt{1+\sigma^2-\sigma^4}}+\dfrac{1-\sigma^2+\sigma^4+2\sqrt{1-\sigma^2}}{1+\sigma^2-\sigma^4}\right]}}{\sqrt{2}(-z^2+1)}\\[2ex] \quad=\frac{1}{2}\sin^{-1}\dfrac{\left(z-\dfrac{\sigma^2-\sqrt{1-\sigma^2}}{\sqrt{1+\sigma^2-\sigma^4}}\right)\sqrt{\left[z^2+2z\dfrac{1-\sqrt{1-\sigma^2}}{\sqrt{1+\sigma^2-\sigma^4}}+\dfrac{1-\sigma^2+\sigma^4-2\sqrt{1-\sigma^2}}{1+\sigma^2-\sigma^4}\right]}}{\sqrt{2}(-z^2+1)},\end{cases}$$

and

$$(23)\qquad \frac{d\psi'}{dz}=\frac{-z^2\dfrac{1-\sigma^2}{2\sqrt{1+\sigma^2-\sigma^4}}-2z\dfrac{\sigma^2\sqrt{1-\sigma^2}}{1+\sigma^2-\sigma^4}+\dfrac{1-\sigma^2}{2\sqrt{1+\sigma^2-\sigma^4}}}{(-z^2+1)\sqrt{Z_1Z_2}}.$$

Replacing σ^2 by k,

$$(24)\quad\begin{cases}\psi'=\frac{1}{2}\cos^{-1}\dfrac{\left(-z-\dfrac{k+\sqrt{1-k}}{\sqrt{1+k-k^2}}\right)\sqrt{\left[z^2-2z\dfrac{1+\sqrt{1-k}}{\sqrt{1+k-k^2}}+\dfrac{1-k+k^2+2\sqrt{1-k}}{1+k-k^2}\right]}}{\sqrt{2}(-z^2+1)}\\[2ex] \quad=\frac{1}{2}\sin^{-1}\dfrac{\left(z-\dfrac{k-\sqrt{1-k}}{\sqrt{1+k-k^2}}\right)\sqrt{\left[z^2+2z\dfrac{1-\sqrt{1-k}}{\sqrt{1+k-k^2}}+\dfrac{1-k+k^2-2\sqrt{1-k}}{1+k-k^2}\right]}}{\sqrt{2}(-z^2+1)},\end{cases}$$

$$(25)\qquad \frac{d\psi'}{dz}=\frac{-z^2\dfrac{1-k}{2\sqrt{1+k-k^2}}-2z\dfrac{k\sqrt{1-k}}{1+k-k^2}+\dfrac{1-k}{2\sqrt{1+k-k^2}}}{(-z^2+1)\sqrt{Z}},$$

$$(26)\qquad L'=\sqrt{\frac{1-k}{1+k-k^2}},\quad L=0.$$

30. With $a = 1$ and a parameter

(1) $$v = \omega_1 + \tfrac{1}{3}\omega_3,$$

(2) $$N_a^2 = B^2 + L^2 + \sigma - s_a = s - s_a, \text{ with } B^2 + L^2 = s - \sigma,$$

and, as in §22,

(3) $$\sigma = 2c, \quad \sigma - s_2 = 2c - 1, \quad \surd(s_1 - \sigma \,.\, \sigma - s_3) = c\surd(2c - 1),$$
$$\surd - \Sigma = 2c(2c - 1), \quad P = \tfrac{1}{3}(2c - 1),$$

so that

(4) $$N_2^2 = s - s_2 = s - 1,$$

(5) $$N_1^2 N_3^2 = (s - s_1)(s - s_3) = s^2 - (c^2 + 2c)s + c^2.$$

As in §21 we can put

(6) $$\frac{s - s_2}{s - \sigma} = q^2, \quad s - \sigma = \frac{2c - 1}{q^2 - 1}, \quad s = \frac{2cq^2 - 1}{q^2 - 1},$$

and then

(7) $$N_2^2 = (2c - 1)\frac{q^2}{q^2 - 1}, \quad N_1^2 N_3^2 = (2c - 1)\frac{-Q_1 Q_2}{(q^2 - 1)^2},$$

with Q_1, Q_2 as in (16), §21; and now for $\mu = 6$, as in §22,

(8) $$e^{\alpha i} = \left[\sqrt{\frac{q + 1}{2}}\surd Q_1 + i\sqrt{\frac{q - 1}{2}}\surd - Q_2\right]e^{\frac{3}{2}\beta i},$$

(9) $$Q_1,\ Q_2 = -cq \pm cq + 1.$$

Now we have

(10) $$\left\{\begin{aligned} \psi &= \tfrac{1}{3}\cos^{-1}\frac{Hz^2 + H_1 z + H_2}{(-z^2 + 1)^{\frac{3}{2}}}\surd(-Z_1) \\ &= \tfrac{1}{3}\sin^{-1}\frac{Kz^2 + K_1 z + K_2}{(-z^2 + 1)^{\frac{3}{2}}}\surd(-Z_2), \end{aligned}\right.$$

(11) $$Z_1 = \left(z - \frac{B + N_2}{M}\right)^2 - \left(\frac{N_1 - N_3}{M}\right)^2, \; Z_2 = \left(z - \frac{B - N_2}{M}\right)^2 - \left(\frac{N_1 + N_3}{M}\right)^2$$

(12) $$M^2 = -L^2 + 2Lc(q^2 - 1) - c^2 + 1 - 2B\frac{q\surd(-Q_1 Q_2)}{\surd(q^2 - 1)}.$$

But with $a = -1$,

(13) $$\psi = \tfrac{1}{3}\cos^{-1}\frac{Hz^2 + H_1 z + H_2}{(-z^2 + 1)^{\frac{3}{2}}}\surd - Z_1$$
$$= \tfrac{1}{3}\sin^{-1}\frac{Kz^2 + K_1 z + K_2}{(-z^2 + 1)^{\frac{3}{2}}}\surd Z_2,$$

with

(14) $$H^2 - K^2 = 1, \quad H = \text{ch}\alpha, \quad K = \text{sh}\alpha,$$

(15) $$-B^2 + L^2 = s - \sigma, \quad B^2 - L^2 = \sigma - s,$$

(16) $$B = \sqrt{s - \sigma}\,\text{sh}\beta, \text{ or } \sqrt{\sigma - s}\,\text{ch}\beta,$$

(17) $$L = \sqrt{s - \sigma}\,\text{ch}\beta, \text{ or } \sqrt{\sigma - s}\,\text{sh}\beta,$$

and

(18) $$e^{\pm\alpha} = \left(\sqrt{\frac{q+1}{2}}\sqrt{Q_1} \pm \sqrt{\frac{q-1}{2}}\sqrt{Q_2}\right)e^{\pm\frac{1}{2}\beta}.$$

The determination of H_1, K_1, H_2, K_2 can then be carried out by (22), (23), §24, and the verification effected, but the presence of M makes the expressions complicated, much more so than for the previous form of solution in §19, so we do not proceed further with this or higher values of n, with z as variable.

31. A similar procedure will determine the leading coefficient H and K when μ is odd, and ψ' is expressed in the form of equation (2), §12, although there is no need to carry out the rest of the calculation now that the degree can be halved by changing to the stereographic projection with the new variable $t = \tan\frac{1}{2}\theta$ in the place of $z = \cos\theta$.

In this case $a = -1$, and taking the parameter

(1) $$v = \frac{4\omega_3}{\mu}, \text{ so that } s(v) = 0,$$

and putting

(2) $$-B^2 + L^2 = s(2w) - s(v) = c^2,$$

we can write

(3) $$I = I(2w, v) = \frac{2}{\mu}\,\text{ch}^{-1}\frac{F_1\sqrt{C_1}}{c^{\frac{1}{2}\mu}} = \frac{2}{\mu}\,\text{sh}^{-1}\frac{F_2\sqrt{C_2}}{c^{\frac{1}{2}\mu}},$$

(4) $$e^{\frac{1}{2}\mu I} = \frac{F_1\sqrt{C_1} + F_2\sqrt{C_2}}{c^{\frac{1}{2}\mu}},$$

(5) $$e^{\mu I} = \frac{F_1\sqrt{C_1} + F_2\sqrt{C_2}}{F_1\sqrt{C_1} - F_2\sqrt{C_2}},$$

(6) $$H + K = \frac{F_1\sqrt{C_1} + F_2\sqrt{C_2}}{2(L+B)^{\frac{1}{2}\mu}},$$

(7) $$-H + K = \frac{F_1\sqrt{C_1} - F_2\sqrt{C_2}}{2(L-B)^{\frac{1}{2}\mu}}.$$

But if $L = B$, $c = 0$, and

$$H,\ K = \frac{F_1^2 xy \mp (2L)^\mu}{(2L)^{\frac{1}{2}\mu} F_1 \sqrt{xy}}, \tag{8}$$

$$\left\{\begin{aligned} M^2 &= L^2 + N_1^2 + N_2^2 + N_3^2 + \frac{B^2\wp''v - \frac{1}{2}\wp'^2 v}{Bi\wp'v} \\ &= L^2 + s_1 + s_2 + s_3 + \frac{\quad\vdots\vdots\vdots\vdots\quad}{} . \end{aligned}\right. \tag{9}$$

Take for example

$$\left\{\begin{aligned} \mu &= 3\,,\ v = \frac{4\omega}{3}\,, \\ S &= 4s^3 - (s - m)^2, \\ s_1 + s_2 + s_3 &= \tfrac{1}{4}\,, \\ s_2s_3 + s_3s_1 + s_1s_2 &= \tfrac{1}{2}m = \tfrac{1}{2}\wp''v\,, \\ s_1s_2s_3 &= \tfrac{1}{4}m^2 = -\tfrac{1}{4}\wp'^2v\,,\ i\wp'v = m, \\ H,\ K &= \frac{m \mp (2L)^3}{(2L)^{\frac{3}{2}}\sqrt{m}}, \\ M^2 &= (L + \tfrac{1}{2})^2 + \frac{m}{2L}\,, \\ Z &= -\left(z - \frac{L}{M}\right)^4 + \frac{1}{2M^2}\left(z - \frac{L}{M}\right)^2 \\ &\qquad - 4\,\frac{m}{M^3}\left(z - \frac{L}{M}\right) - \frac{1 - 32m}{16M^4}\,; \end{aligned}\right. \tag{10}$$

or, since

$$m = 2LM^2 - 2L(L + \tfrac{1}{2})^2 \tag{11}$$

$$\left\{\begin{aligned} Z = &-\left(z - \frac{L}{M}\right)^4 + \frac{1}{2M^2}\left(z - \frac{L}{M}\right)^2 \\ &- \frac{\delta LM^2 - \delta L(L + \frac{1}{2})^2}{M^2}\left(z - \frac{L}{M}\right) \\ &- \frac{1}{16M^4} + \frac{4LM^2 - 4L(L + \frac{1}{2})^2}{M^4}. \end{aligned}\right. \tag{12}$$

Putting $z = \pm 1$,

$$\left\{\begin{aligned} Z &= -4\left(\frac{L' \mp L}{M}\right)^2 \\ &= -\left(\frac{3L^2 + 2L + \frac{1}{4}}{M^2} \mp 2\,\frac{L}{M} - 1\right)^2, \end{aligned}\right. \tag{13}$$

so that

(14) $$2\frac{L'}{M}=\frac{3L^2+2L+\frac{1}{4}}{M^2}-1$$

(15) $$L'+L=\frac{(3L+\frac{1}{2}-M)\,(M+L+\frac{1}{2})}{2M},$$

$$L-L'=\frac{(3L+\frac{1}{2}+M)\,(M-L-\frac{1}{2})}{2M},$$

(16) $$Z+4\left(\frac{Lz-L'}{M}\right)^2=-(z^2-1)\left(z^2-4\frac{L}{M}z-1+E\right)$$

(17) $$E=\frac{4L^2-1}{2M^2}+2.$$

In the result of §14 we have now to take

(18) $$\gamma=-\tfrac{1}{2},\ k=\frac{3L+\frac{1}{2}+M}{3L+\frac{1}{2}-M}\sqrt{\frac{M-L-\frac{1}{2}}{M+L+\frac{1}{2}}}.$$

Try

(19) $$m=-(2L)^3,\ K=0,$$

(20) $$M^2=\tfrac{1}{4}(1+6L)\,(1-2L)\ .$$

(21) $$4M-1-4L=-(\sqrt{1+6L}-\sqrt{1-2L})^2$$

(22) $$4M+1+4L=\ \ (\sqrt{1+6L}+\sqrt{1-2L})^2,$$

so that k is imaginary.

Try

(23) $$m=(2L)^3,\ H=0,$$

(24) $$M^2=(L+\tfrac{1}{2})^2+4L^2$$

(25) $$4M^2-(2L+1)^2=16L^2$$

(26) $$\lambda^2=k=\left(\frac{\sqrt{2M+2L+1}+\sqrt{2M-2L-1}}{\sqrt{2L+1}}\right)^2.$$

Thus, as a numerical case, with

(27) $$L=1,\ M=\frac{5}{2},\ m=8,\ L'=-\frac{1}{5},\ k=3,$$

(28) $$Z=-z^4+\frac{8}{5}z^3-\frac{22}{25}z^2-\frac{232}{125}z+\frac{759}{625},$$

$Z=0$ having two real roots between ± 1; and we find

$$(29)\qquad \psi=\tfrac{1}{3}\cos^{-1}\frac{\frac{7}{5}z^2+\frac{4}{5}z-\frac{59}{125}}{(\sqrt{-z^2+1})^{\frac{3}{2}}}=\tfrac{1}{3}\sin^{-1}\frac{\left(z+\frac{4}{5}\right)\sqrt{Z}}{(-z^2+1)^{\frac{3}{2}}},$$

or changing to the stereographic projection,

$$(30)\qquad 10\sqrt{5}(te^{\psi' i})^{\frac{1}{2}}=(t-3)\sqrt{(4t^4+24t^3-5t^2+4t+6)}$$
$$+i(t+3)\sqrt{(-4t^4+24t^3+5t^2+4t-6)}.$$

For the motion of the centre

$$(31)\qquad \rho^2=\left(z-\frac{4}{5}\right)^2+\frac{3}{5},$$

and

$$(32)\qquad \rho^3e^{3\omega' i}=z^3-\frac{12}{5}z^2+H_2z+H_3+iK_1\sqrt{Z},$$

$$(33)\qquad K_1=1,\ H_2=\frac{83}{25},\ H_3=-\frac{104}{125}.$$

The degree is halved by putting

$$(34)\qquad z-\frac{4}{5}=\sqrt{\frac{3}{5}}\frac{2w}{w^2-1},\ \rho=\sqrt{\frac{3}{5}}\frac{w^2+1}{w^3-1},$$

and then

$$(35)\qquad (\rho e^{\omega' i})^{\frac{3}{2}}=\frac{(w+1)\sqrt{W_1}+i(w-1)\sqrt{W_2}}{\sqrt[4]{(500)}\,(w^3-1)^{\frac{3}{2}}},$$

$$(36)\qquad W_1=(4\sqrt{5}+3\sqrt{3})w^4-8(\sqrt{5}-\sqrt{3})w^3-10\sqrt{3}w^2+8(\sqrt{5}+\sqrt{3})w$$
$$-4\sqrt{5}+3\sqrt{3},$$

$$(37)\qquad W_2=-(4\sqrt{5}-3\sqrt{3})w^4-8(\sqrt{5}+\sqrt{3})w^3-10\sqrt{3}w^2+8(\sqrt{5}-\sqrt{3})w$$
$$+4\sqrt{5}+3\sqrt{3}.$$

A numerical application, such as inserted here and elsewhere, is useful in showing the existence of a real solution, and the region of its reality, as well as in settling a doubtful sign in the general algebraical case.

The Theory of Implication.

By Bertrand Russell.

The purpose of the present article is to set forth the first chapter of the deduction of pure mathematics from its logical foundations. This first chapter is necessarily concerned with deduction itself, *i. e.* with the principles by which conclusions are inferred from premisses. If it is our purpose to make all our assumptions explicit, and to effect the deduction of all our other propositions from these assumptions, it is obvious that the first assumptions we need are those that are required to make deduction possible. Symbolic logic is often regarded as consisting of two coordinate parts, the theory of classes and the theory of propositions. But from our point of view these two parts are not coordinate; for in the theory of classes we deduce one proposition from another by means of principles belonging to the theory of propositions, whereas in the theory of propositions we nowhere require the theory of classes. Hence, in a deductive system, the theory of propositions necessarily precedes the theory of classes.

But the subject to be treated in what follows is not quite properly described as the theory of *propositions*. It is in fact the theory of how one proposition can be inferred from another. Now in order that one proposition may be inferred from another, it is necessary that the two should have that relation which makes the one a consequence of the other. When a proposition q is a consequence of a proposition p, we say that p *implies* q. Thus deduction depends upon the relation of *implication*, and every deductive system must contain among its premisses as many of the properties of implication as are necessary to legitimate the ordinary procedure of deduction. In the present article, certain propositions concerning implication will be stated as premisses, and it will be shown that they are sufficient for all common forms of inference. It will not be shown that they are all *necessary*, and it is probable that the number of them might be diminished. All that is affirmed concerning the premisses is (1) that they are true, (2) that they are sufficient for the theory of deduction, (3) that I do not know how to diminish their number. But with regard to (2), there must always be some element of doubt, since it is hard to be sure that one never uses some principle unconsciously. The habit of being rigidly guided by formal symbolic rules is a safeguard against unconscious assumptions; but even this safeguard is not always adequate.

The symbolism adopted in what follows is that of Peano, with certain additions and changes. I have adopted his symbol for implication ($\supset$), and his use of dots instead of brackets; also his plan of numbering propositions with an integral and a decimal part.* But although the symbolism is in the main Peano's, the ideas are more those of Frege.† Frege's work, probably owing to the inconvenience of his symbols, has received far less recognition than it deserves. I shall not refer to him in detail in what follows, but whoever will consult his work will see how much I owe to him. Especially I have adopted from him the assertion-sign (cf. ★1·1 *infra*), the interpretation of "p implies q" (cf ★1·2 *infra*), and the distinction between asserting a proposition for *all* values of the variable or variables, and asserting it for any values (cf. ★7 *infra*). The plan of taking implication and negation as our primitive ideas is also his.

★1. *Primitive Ideas.*

Since all definitions of terms are effected by means of other terms every system of definitions which is not circular must start from a certain apparatus of undefined terms. It is to some extent optional what ideas we take as undefined in mathematics; the motives guiding our choice will be (1) to make the number of undefined ideas as small as possible, (2) as between two systems in which the number is equal, to choose the one which seems the simpler and easier. I know no way of proving that such-and-such a system of undefined ideas contains as few as will give such-and-such results.‡ Hence we can only say that such and such ideas are undefined in such and such a system, not that they are indefinable. Following Peano, I shall call the undefined ideas and the undemonstrated propositions *primitive* ideas and propositions respectively. These ideas are *explained* by means of descriptions intended to point out to the reader what is meant; but the explanations do not constitute definitions, because they really involve the ideas they explain.

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* For an explanation of Peano's symbolism, cf. Whitehead, "On Cardinal Numbers," *American Journal of Mathematics*, Vol. XXIV, No. 4.

† Cf. his "Grundgesetze der Arithmetik," Vol. I, Jena, 1893; Vol. II, 1903.

‡ The recognized methods of proving independence are not applicable, without reserve, to fundamentals. Cf. my "Principles of Mathematics", §17. What is there said concerning primitive propositions applies with even greater force to primitive ideas.

★1·1. *Assertion.*—Any proposition may be either asserted or merely considered. If I say "Caesar died", I assert the proposition "Caesar died" if I say "'Caesar died' is a proposition", I make a different assertion, and "Caesar died" is no longer asserted, but merely considered. Similarly in a hypothetical proposition, *e. g.* "if $a = b$, then $b = a$", we have two unasserted propositions, namely "$a = b$" and "$b = a$" while what *is* asserted is that the first of these implies the second. In language, we indicate when a proposition is merely considered by "*if* so-and-so" or "*that* so-and-so" or merely by inverted commas. In symbols, if p is a proposition, p by itself will stand for the unasserted proposition, while the asserted proposition will be designated by

$$\text{“}\vdash . p\text{”}$$

The sign "$\vdash$" is called the assertion-sign; it may be read "it is true that" (although philosophically this is not what it means). The dots after the assertion-sign indicate its range; that is to say, everything following is asserted until we reach either an equal number of dots or the end of the sentence. Thus "$\vdash : p . \supset . q$" means "it is true that p implies q", whereas "$\vdash . p . \supset \vdash . q$" means "$p$ is true; therefore q is true". The first of these does not necessarily involve the truth either of p or of q, while the second involves the truth of both.

★1·2. *Implication.*—The meaning to be given to implication in what follows may at first sight appear somewhat artificial; but although there are other legitimate meanings, the one here adopted is, if I am not mistaken, very much more convenient that any of its rivals. The essential property that we require of implication is this: "What is implied by a true proposition is true". It is in virtue of this property that implication yields proofs. But this property by no means determines whether anything, and if so what, is implied by a false proposition, or by something which is not a proposition at all. What it does determine is that, if p

implies q, then it cannot be the case that p is true and q is not true. The most convenient interpretation of implication is to say, conversely, that unless p is true and q is not true, "p implies q" is to be true. Hence, "p implies q" will be a relation which holds between any two entities p and q unless p is true and q is not true, *i. e.* whenever either p is not true or q is true.[*] The proposition "p implies q" is equivalent to "if p is true, then q is true", *i. e.* "'p is true' implies 'q is true'"; it is also equivalent to "if q is false, p is false". When p is in fact true, "implies" may be replaced by "therefore", *i. e.* in place of "p implies q" we may say "p is true; therefore q is true". For "implies" we use the symbol "$\supset$", thus

"$p \supset q$" means "p implies q"
"$p . \supset . q \supset r$" means "$p$ implies that q implies r", etc.

The chief advantage of the above interpretation is that it avoids hypotheses which are otherwise necessary. We wish, for example, to assert "$p \supset p$". If implication can only hold between propositions, it is necessary to preface "$p \supset p$" by the hypothesis "p is a proposition"; and whenever we want to use "$p \supset p$" in a particular case, we shall have to prove first that what we are applying it to is a proposition.[†] But this is highly inconvenient. Again, paradoxes result from restricting the meaning of implication. For example, it will be admitted that "if p and q are true, then r is true" is equivalent to "if p is true, then if q is true, r is true", *i. e.* to "if p is true, then q implies r". Also it will be admitted that if p and q are true, then p is true. Hence, by the above admission, if p is true, then q implies p; and here q is not subject to any limitation; for, even if q is not a proposition, it must be admitted that if p and q *were* both true, p *would be* true. Hence, unless a true proposition p is to be implied by *every* entity q, one at least of the above obvious propositions will have to be denied.

★1·3. *The Variable.*—A single letter, unless specially defined to have a certain constant meaning, will always stand for an independent variable. The possible values of an independent variable are always to include all entities absolutely. The reason for this is as follows: If we affirm some statement about x, where x is restricted by some condition, we must mention the condition to make our statement accurate; but then we are really affirming that the truth of the condition implies the truth of our original statement about x; and this, in virtue of our interpretation of implication, will hold equally when the condition is not fulfilled. The "universe of discourse", as it has been called, must be replaced by a general hypothesis concerning the variable, and then our formulæ are true whether the hypothesis is verified or not, because an implication holds whenever its hypothesis is not true. The old theory of the "universe" had the defect of introducing tacit hypotheses, thus making all enunciations incomplete, since a hypothesis does not cease to be an essential part of a proposition merely because we do not take the trouble to state it.

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[*] Cf. ★5·53 and ★5·54 below.

[†] In my "Principles of Mathematics" I adopted the interpretation "p and q are propositions, and p is *false* or q is true", instead of, as here, "p is *not true* or q is true". Thus "$p \supset q$" was false if p was not a proposition, and "$p \supset p$" was equivalent to "p is a proposition". The advantages of the present interpretation may be seen by comparing the primitive propositions of ★2 below with those of §18 of the above-mentioned work.

In such a formula as (say) "$x . = . p \supset q$", the x is still an *independent* variable, because we are not only concerned with the value of x that makes the formula *true*, but also with all the other values that make it false. On the other hand, "$p \supset q$" is a *dependent* variable.

★1·4. *Propositional Functions.*—Any statement about a variable x will be expressed by

$$(C \between x)$$

or by $(A \between x)$, $(B \between x)$, etc. Similarly any statement about two variables x and y will be expressed by

$$(C \between x, y)$$

or by $(A \between x, y)$, $(B \between x, y)$ etc., and so on for any number of variables. Such expressions are functions whose values are propositions; hence we call them *propositional functions*. Thus "$p \supset q$" is a propositional function of p and q; "$p \supset p$" is a propositional function of p. There are other kinds of functions in mathematics, but they are not required for the theory of implication.

When an expression of the form $(C \between x)$ is asserted, what is meant is that the expression in question is true for *any* value of the variable x. Thus, *e. g.*

$$\text{“}\vdash . p \supset p\text{”}$$

means: "For *any* value of p, p implies p". But such an expression as

$(C \between p \supset q)$ does not assert that $(C \between x)$ holds for *any* argument, but only for such as are of the form $p \supset q$; *e. g.* if $(C \between x)$ is "x is a proposition", then

"$\vdash . (C \between x)$" is false, but
"$\vdash . (C \between p \supset q)$" is true.

Generally, "$\vdash . (C \between (A \between y))$" means: "For any value of y, $(C \between (A \between y))$ is true", or "For any value of y, $(A \between y)$ is a value of x for which $(C \between x)$ is true". The C in $(C \between x)$ has no meaning by itself; all that its use assumes is that $(C \between x)$ and $(C \between y)$ are to have similarity of *form*, although this form is not always capable of separate specification apart from any argument x.

★1·5. *Negation.*—The proposition "p is not true" is expressed by

$$\sim p.$$

Thus "$\sim p$" will be true if p is not a proposition, and if p is a false proposition; it will only be false if p is a true proposition.

One other primitive idea will be introduced in ★7. Two others, or perhaps three, complete what is required for the whole of pure mathematics. Thus, six are necessary for the theory of implication, and eight or nine for all pure mathematics.

★2. *Primitive Propositions.*

·1 Anything implied by a true proposition is true. Pp.[*] This proposition is used in every proof without exception; hence it will not be referred to in giving proofs. The very meaning of *proof* is demonstration of truth by the fact of being implied by a true proposition. It must be understood that the principle is not the same as: "*If* p is true, then *if* p implies q, q is true". This is a true proposition, but it holds equally when p is not true, and when p does not imply q. It does not, like the principle we are concerned with, enable us to assert q simply, without any hypothesis. We cannot express the principle symbolically, partly because any symbolism in which p is variable only gives the *hypothesis* that p is true, not the *fact* that it is true.[†]

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[*] The letters "Pp" stand for "primitive proposition" as with Peano.

[†] For further remarks on this principle, cf. my "Principles of Mathematics", §38.

When this principle is used, we write

$$\text{“}\vdash . p . \supset \vdash . q\text{”}$$

meaning "p, which is true, implies q, which is therefore true", or "p is true; therefore q is true". *Therefore* is distinguished from *implies* by being only applicable to implications between *true* propositions.

★2·2. If a propositional function $(C \between y)$ is true for *any* value of y, it is true for such-and-such a value. Pp.

This principle will be called the "principle of substitution", and will be referred to as "Subst." It allows us, for example, from "For any value of y, if y is a man, y is a mortal", to infer "if Socrates is a man, Socrates is a mortal". The value substituted is supposed to be a definite constant value, such as Socrates, or the fourth proposition of Euclid. This principle, like its predecessor, cannot be symbolized, because any possible symbolic expression would deal with *any* constant, and we should need the principle itself to deduce, from such a statement, that it holds also concerning Socrates or some other definite constant. "Any constant" is, in fact, not distinguishable from a variable: it is like what is called a parameter, *i. e.* a variable which we choose to regard as a constant.

★2·3. If a propositional function $(C \between y)$ is true for any value of y, then $(C \between (A \between z))$ is true for any value of z. Pp.

This may be called the "principle of the substitution of a dependent for an independent variable". It may be referred to as "Dep.", where "Dep." is short for "dependent variable". It is obvious that, since all our variables have an unrestricted range, the various values of $(A \between z)$ are among the values of y, since all entities are among the values of y. Hence what holds for any value of y holds also for any value of $(A \between z)$. [This is not a proof, but merely an elucidation.]

To symbolize this principle, we should need a symbol for the supposition, as opposed to the assertion, that $(C \between y)$ is true for *any* value of y. Such a symbol is not afterwards needed, since our principle will only be applied when $(C \between y)$ is *asserted* for any value of y. In any such case the principle becomes

$$\vdash . (C \between y) . \supset \vdash . (C \between (A \between z)).$$

The use of this principle is constant. For example, it is needed for the inference that, since "$p \supset p$" is true for any value of p, it is true if we substitute $p \supset q$ for p, so that "$p \supset q \,.\supset.\, p \supset q$" is true for any values of p and q. (The principle is to be extended to two or more variables.) When this principle is used, the substitution effected will be indicated by

$$(C \between y)\,\frac{(A \between z)}{y}$$

meaning that, in $(C \between y)$, $(A \between z)$ is to be substituted for y. Generally the substitution is to be effected in a proposition referred to by its name or number. Thus, *e. g.* "$\vdash .\, p \supset p$" will be $\bigstar\, 2\cdot5$; thus,

$$\text{“}\bigstar\, 2\cdot5\, \frac{p \supset q}{p}\text{”}$$

means what $\bigstar\, 2\cdot5$ becomes when "$p \supset q$" is substituted for p. Thus we write

$$\vdash .\, \bigstar\, 2\cdot5\, \frac{p \supset q}{p} \,.\supset \vdash : p \supset q \,.\supset.\, p \supset q$$

in which we use our present principle. In like manner,

$$(C \between x, y)\,\frac{(A \between x, y),\ (B \between x, y)}{x, y}$$

means the result of substituting $(A \between x, y)$, $(B \between x, y)$ for x, y in $(C \between x, y)$. Thus $\bigstar\, 2\cdot6$ will be "$\vdash : p \,.\supset.\, q \supset p$", *i. e.* "If p is true, then, if q is true, p is true". Hence

$$\text{“}\bigstar\, 2\cdot6\, \frac{p \supset q \,.\supset.\, q,\ q \supset p}{p, q}\text{”}$$

denotes the result of substituting $p \supset q \,.\supset.\, q$ for p, and $q \supset p$ for q: thus we may write

$$\vdash .\, \bigstar\, 2\cdot6\, \frac{p \supset q \,.\supset.\, q,\ q \supset p}{p, q} \,.$$

$$\supset \vdash :: p \supset q \,.\supset.\, q :\supset:. \, q \supset p \,.\supset: p \supset q \,.\supset.\, q$$

[In reading expressions of this kind, the first thing to notice is the arrangement of the dots, which gives the structure of the sentence. Thus the second line of the above reads: "If 'p implies q' implies q, then, if q implies p, it follows that 'p implies q' implies q".]

In like manner, when a constant, say Socrates, is to be substituted to obtain an inference by ★2·2, we may indicate the inference by

$$(C\,)(\,y)\,\frac{\text{Socrates}}{y}.$$

Thus, $\vdash . \bigstar 2{\cdot}6\,\frac{\text{Socrates}}{q} . \supset \vdash : p . \supset . \text{Socrates} \supset p .$

But this is less often necessary, since the substitution of a constant is generally easily recognizable.

The remaining principles of deduction are simpler. They are as follows:

★2·5 $\vdash . p \supset p$ Pp. ["Id"][*]

This principle asserts that anything implies itself. We decided that "$p \supset q$" was to be equivalent to "p is not true or q is true"; hence "$p \supset p$" is equivalent to "p is not true or p is true", *i. e.* to the law of excluded middle. This principle will be referred to as "Id", which is short for "identity"; for the principle is a form of the law of identity.

★2·6 $\vdash : p . \supset . q \supset p$ Pp. ["Simp"]

This principle asserts that if p is true, then if q is true, p is true; it is equivalent to "if p and q are true, then p is true". On this account it is called (following Peano) the "principle of simplification", which is abbreviated into "Simp." This principle asserts that a true proposition is implied by anything, for it may be read: "If p is true, then q implies p".

★2·7 $\vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$ Pp. ["Syll"]

This principle asserts: "If p implies q, then, if q implies r, it follows that p implies r". This principle is called the "principle of the syllogism" (shortened to "Syll"), because it is the source of the syllogism in Barbara.[†]

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[*] A name in inverted commas, enclosed in square brackets, is the name of the proposition after which it occurs. I have given names to a few of the more important principles, for convenience of reference. When there are no inverted commas, the enclosure in square brackets refers to the proposition or propositions by which the one in question is proved.

[†] This occurs, in outline, as follows:

"All a is b" is equivalent to "for any value of x, x is an a . $\supset$. x is a b".

"All b is c" is equivalent to "for any value of x, x is a b . $\supset$. x is a c".

By the above principle, if these both hold, it follows that for any value of x, x is an a . $\supset$. x is a c, *i. e.* "All a is c". The complete proof is more elaborate, but the above is only intended to justify the name.

★2·8 $\vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$ Pp. ["Comm"]

i. e., given that, if p is true, then, if q is true, r is true, it follows that, if q is true, then, if p is true, r is true. Roughly, if r follows from p and q, then r follows from q and p. This is called "the commutative principle" (shortened to "Comm").

·9 $\vdash . \sim(\sim p) \supset p$ Pp. ["Neg"]

i. e., if it is false that p is false, then p is true. This is the "principle of double negation" (shortened to "Neg").

·91 $\vdash : p \supset \sim p . \supset . \sim p$ Pp. ["Abs"]

i. e., whatever implies its own untruth is untrue. This principle is very useful in philosophy, where many widely-held positions sin against it. *E. g.* the principle "no truth is quite true" implies that itself is quite true, and therefore that some truth is quite true. Hence it implies its own falsehood, and is therefore false. It is called the "principle of the reductio ad absurdum" (shortened into "Abs").*

★2·92 $\vdash : p \supset \sim q . \supset . q \supset \sim p$ Pp. ["Transp"]

i. e., if the truth of p implies the untruth of q, then the truth of q implies the untruth of p. In other words, if p is incompatible with q, then q is incompatible with p. This is called the "principle of transposition" (shortened to "Transp").

In ★7, three more primitive propositions will be given. For the present the above completes the list. In what follows, a few of the more important propositions will be given names, but the rest will be referred to by their numbers.

★3. *Elementary Properties of Implication and Negation.*

The propositions that follow are all such as are actually needed in deducing mathematics from our primitive propositions. I shall omit proofs of some of the less important, and shall sometimes only briefly indicate the proofs where they are very obvious. But in most cases I shall give the proofs in full,† because the importance of the present subject lies, not in the propositions themselves, but (1) in the fact that they follow from the primitive propositions, (2) in the fact that it is the easiest, simplest, and most elementary example of the symbolic method of dealing with the principles of mathematics generally. Later portions —the theories of classes, relations, cardinal numbers, series, ordinal numbers, geometry, etc.—all employ the same method, but with an increasing complexity in the entities and functions considered.

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* There is an interesting historical article on this principle by Vailati, "A proposito d' un passo del Teeteto e di una dimostrazione di Euclide", Rivista di Filosofia e scienze affine, 1904.

† Certain abbreviating processes will, however, be gradually introduced.

★3·1 $\vdash :. p . \supset : p \supset q . \supset . q$

Dem.

$$\vdash . \text{Id}\,\frac{p \supset q}{p} . \supset \vdash : p \supset q . \supset . p \supset q \tag{1}$$

$$\vdash . \text{Comm}\,\frac{p \supset q, p, q}{p, q, r} . \supset \vdash :: p \supset q . \supset . p \supset q : \supset :. p . \supset : p \supset q . \supset . q \tag{2}$$

$\vdash . (1) . \supset : \vdash . (2) . \supset \vdash . \text{Prop.}$

In the above, *Dem.* is short for *demonstration.* The proposition proved may be stated: "If p is true, then, if p implies q, q is true". This differs from ★2·1 by the fact that it is not only concerned with the case when p actually is true, and does not end by asserting that q actually is true. The symbol "$\text{Id}\,\frac{p \supset q}{p}$" has been explained in connection with ★2·3. The passage from (1) and (2) to the conclusion is effected in accordance with ★2·1; (1) asserts the hypothesis of (2), and the apodosis in (2) is the proposition to be proved. In future, instead of "$\vdash . (1) . \supset : \vdash . (2) . \supset \vdash . \text{Prop.}$", I shall write "$\vdash . (1) . (2) . \supset \vdash . \text{Prop.}$". The numbers (1), (2), etc. refer to everything to the right of the last assertion-sign in the line in which they occur. "Prop." always means the proposition to be proved; thus every proof ends with "$\supset \vdash . \text{Prop.}$", which takes the place of "Q. E. D."

★3·11 $\vdash : q . \supset . p \supset p$

Dem.

$$\vdash . \text{Comm}\,\frac{p}{r} . \supset \vdash :. p . \supset . q \supset p : \supset : q . \supset . p \supset p \tag{1}$$

$\vdash . (1) . \text{Simp.} \supset \vdash . \text{Prop.}$

·12 $\vdash :. q \supset r . \supset : p \supset q . \supset . p \supset r$

Dem.

$\vdash . \text{Comm}\,\frac{p \supset q, q \supset r, p \supset r}{p, q, r} . \supset$

$$\vdash :: p \supset q . \supset : q \supset r . \supset . p \supset r :. \supset :. q \supset r . \supset : p \supset q . \supset . p \supset r \tag{1}$$

$\vdash . (1) . \text{Syll.} \supset \vdash . \text{Prop.}$

$\bigstar 3{\cdot}2 \quad \vdash . q \supset \sim(\sim q)$

Dem.

$\vdash . \text{Transp} \dfrac{\sim q}{p} . \supset \vdash : \sim q \supset \sim q . \supset . q \supset \sim(\sim q)$ (1)

$\vdash . (1) . \text{Id.} \quad \supset \vdash . \text{Prop.}$

The above is the converse of the principle of double negation ($\bigstar 2{\cdot}9$); it asserts that if p is true, the negation of p is false.

$\cdot 21 \quad \vdash : \sim p \supset q . \supset . \sim q \supset p$

Dem.

$\vdash . \bigstar 3{\cdot}12 \dfrac{\sim p, \sim(\sim q)}{p, r} . \supset \vdash :. q \supset \sim(\sim q) . \supset : \sim p \supset q . \supset . \sim p \supset \sim(\sim q)$ (1)

$\vdash . \bigstar 3{\cdot}2 \dfrac{q}{p} \quad . \supset \vdash . q \supset \sim(\sim q)$ (2)

$\vdash . (1) . (2) . \quad \supset \vdash : \sim p \supset q . \supset . \sim p \supset \sim(\sim q)$ (3)

$\vdash . \bigstar 2{\cdot}92 \dfrac{\sim p, \sim q}{p, q} \quad . \supset \vdash : \sim p \supset \sim(\sim q) . \supset . \sim q \supset \sim(\sim p)$ (4)

$\vdash . \bigstar 3{\cdot}12 \dfrac{\sim q, \sim(\sim p), p}{p, q, r} . \supset \vdash :. \sim(\sim p) \supset p . \supset : \sim q \supset \sim(\sim p) . \supset . \sim q \supset p$ (5)

$\vdash . (5) . \bigstar 2{\cdot}9 . \quad \supset \vdash : \sim q \supset \sim(\sim p) . \supset . \sim q \supset p$ (6)

$\vdash . (3) . (4) . (6) . \text{Syll.} \quad \supset \vdash . \text{Prop.}$

Note. In the last line of the above proof, "(3) . (4) . (6) . Syll." is an abbreviation for a process which occurs constantly, and which is tedious to write out. It will be observed that (3) is of the form $\vdash . a \supset b$, (4) is of the form $\vdash . b \supset c$, and (6) is of the form $\vdash . c \supset d$, while the proposition to be proved is $a \supset d$. The process of proof, in full, is as follows:

$\vdash . \text{Syll.} \quad \supset \vdash :. a \supset b . \supset : b \supset c . \supset . a \supset c$ (7)

$\vdash . (3) . (7) . \quad \supset \vdash : b \supset c . \supset . a \supset c$ (8)

$\vdash . (4) . (8) . \quad \supset \vdash . a \supset c$ (9)

$\vdash . \text{Syll.} \quad \supset \vdash :. a \supset c . \supset : c \supset d . \supset . a \supset d$ (10)

$\vdash . (9) . (10) . \supset \vdash : c \supset d . \supset . a \supset d$ (11)

$\vdash . (6) . (11) . \supset \vdash . a \supset d$

and "$a \supset d$" was the proposition to be proved. [In the above, a is $\sim p \supset q$, b is $\sim p \supset \sim(\sim q)$, c is $\sim q \supset \sim(\sim p)$, d is $\sim q \supset p$.]

The above proposition is a second form of "Transposition"; the two following (★3·22 and ★3·23) are two further forms. All four may sometimes be referred to as "Transp".

★3·22 $\vdash : p \supset q . \supset . \sim q \supset \sim p$

Dem.

$\vdash . \star 3{\cdot}12 \dfrac{\sim(\sim q)}{r} . \supset \vdash :. q \supset \sim(\sim q) . \supset : p \supset q . \supset . p \supset \sim(\sim q)$ (1)

$\vdash . (1) . \star 3{\cdot}2 . \quad \supset \vdash : p \supset q . \supset . p \supset \sim(\sim q)$ (2)

$\vdash . \star 2{\cdot}92 \dfrac{\sim q}{q} . \quad \supset \vdash : p \supset \sim(\sim q) . \supset . \sim q \supset \sim p$ (3)

$\vdash . (2) . (3) . \mathrm{Syll.} \supset \vdash . \mathrm{Prop.}$

·23 $\vdash : \sim q \supset \sim p . \supset . p \supset q$

Dem.

$\vdash . \star 2{\cdot}92 \dfrac{\sim q , p}{p , q} . \quad \supset \vdash : \sim q \supset \sim p . \supset . p \supset \sim(\sim q)$ (1)

$\vdash . \star 3{\cdot}12 \dfrac{\sim(\sim q) , q}{q , r} . \supset \vdash :. \sim(\sim q) \supset q . \supset : p \supset \sim(\sim q) . \supset . p \supset q$ (2)

$\vdash . (2) . \star 2{\cdot}9 . \quad \supset \vdash : p \supset \sim(\sim q) . \supset . p \supset q$ (3)

$\vdash . (1) . (3) . \mathrm{Syll.} \quad \supset \vdash . \mathrm{Prop.}$

★3·3 $\vdash : p . \supset . \sim p \supset q$

i. e., if p is true, then $\sim p$ implies q, whatever q may be.

Dem.

$\vdash . \mathrm{Simp} \dfrac{\sim q}{q} . \quad \supset \vdash : p . \supset . \sim q \supset p$ (1)

$\vdash . \star 3{\cdot}21 \dfrac{q , p}{p , q} . \quad \supset \vdash : \sim q \supset p . \supset . \sim p \supset q$ (2)

$\vdash . (1) . (2) . \mathrm{Syll.} \supset \vdash . \mathrm{Prop.}$

·31 $\vdash : \sim p . \supset . p \supset q$

i. e., if p is not true, p implies q, whatever q may be.

Dem.

$\vdash . \mathrm{Comm.} \dfrac{\sim p , q}{q , r} . \supset \vdash :. p . \supset . \sim p \supset q : \supset : \sim p . \supset . p \supset q$ (1)

$\vdash . (1) . \star 3{\cdot}3 . \quad \supset \vdash . \mathrm{Prop.}$

·32 $\vdash : \sim(p \supset q) . \supset . p$ [★3·31·21]

Note. Here ★3·31·21 is short for ★3·31·★3·21. When a proof is very obvious, as in the present case, I shall merely indicate the propositions employed in it by enclosing the references in square brackets, as above.

$\cdot 33$ $\vdash : \sim (p \supset q) . \supset . \sim q$ [Simp. $\star 3{\cdot}22$]

$\star 3{\cdot}34$ $\vdash : \sim (p \supset q) . \supset . p \supset \sim q$

Dem.

$\vdash . \text{Simp} \frac{\sim q , p}{p , q} . \supset \vdash : \sim q . \supset . p \supset \sim q$ (1)

$\vdash . (1) . \star 3{\cdot}33 . \text{Syll.} \supset \vdash$ Prop.

$\cdot 35$ $\vdash : \sim (p \supset q) . \supset . \sim p \supset q$ [$\star 3{\cdot}32{\cdot}3$]

$\cdot 36$ $\vdash : \sim (p \supset \sim q) . \supset . p$ $\left[\star 3{\cdot}32 \frac{\sim q}{q}\right]$

$\cdot 37$ $\vdash : \sim (p \supset \sim q) . \supset . q$ $\left[\star 3{\cdot}33 \frac{\sim q}{q} . \star 2{\cdot}9\right]$

$\cdot 38$ $\vdash :. p . \supset : q . \supset . \sim (p \supset \sim q)$

Dem.

$\vdash . \star 3{\cdot}1 \frac{\sim q}{q} . \supset \vdash :. p . \supset : p \supset \sim q . \supset . \sim q$ (1)

$\vdash . \star 2{\cdot}92 \frac{p \supset \sim q}{p} . \supset \vdash :. p \supset \sim q . \supset . \sim q : \supset : q . \supset . \sim (p \supset \sim q)$ (2)

$\vdash . (1) . (2) . \text{Syll.} \supset \vdash$ Prop.

Note. It follows from $\star 3{\cdot}36{\cdot}37$ that when $\sim (p \supset \sim q)$ is true, p and q are true, and it follows from $\star 3{\cdot}38$ that when p and q are true, $\sim (p \supset \sim q)$ is true. This justifies the definition below ($\star 4{\cdot}1$) of the joint assertion or propositional product of p and q.

$\star 3{\cdot}4$ $\vdash :. p \supset q . \supset . r : \supset : p \supset q . \supset . p \supset r$

Dem.

$\vdash . \star 3{\cdot}1 \frac{p \supset q , r}{p , q} . \supset \vdash :: p \supset q . \supset :. p \supset q . \supset . r : \supset . r$ (1)

$\vdash . \star 3{\cdot}12 \frac{p \supset q . \supset . r , r , p \supset r}{p , q , r} . \supset$

$\vdash ::. r . \supset . p \supset r : \supset :: p \supset q . \supset . r : \supset . r :. \supset :. p \supset q . \supset . r : \supset . p \supset r$ (2)

$\vdash . (2) . \text{Simp.} \quad \supset \vdash :: p \supset q . \supset . r : \supset . r :. \supset :. p \supset q . \supset . r : \supset . p \supset r$ (3)

$\vdash . (1) . (3) . \text{Syll.} \supset \vdash :: p \supset q . \supset :. p \supset q . \supset . r : \supset . p \supset r$ (4)

$\vdash . \text{Comm} \frac{p \supset q , p \supset q . \supset . r , p \supset r}{p , \quad q , \quad r} . \supset$

$\vdash ::. p \supset q . \supset :. p \supset q . \supset . r : \supset . p \supset r :: \supset :: p \supset q . \supset . r : \supset :$
$p \supset q . \supset . p \supset r$ (5)

$\vdash . (4) . (5) . \supset \vdash .$ Prop.

★3·41 $\vdash : \sim p \supset p . \supset . p$

Dem.

$\vdash . \star 3{\cdot}12 \dfrac{\sim p , p , \sim(\sim p)}{p , q , r} . \supset \vdash :. p \supset \sim(\sim p) . \supset : \sim p \supset p .$

$\supset . \sim p \supset \sim(\sim p)$ (1)

$\vdash . (1) . \star 3{\cdot}2 . \quad \supset \vdash : \sim p \supset p . \supset . \sim p \supset \sim(\sim p)$ (2)

$\vdash . \star 2{\cdot}91 \dfrac{\sim p}{p} . \supset \vdash : \sim p \supset \sim(\sim p) . \supset . \sim(\sim p)$ (3)

$\vdash . \star 2{\cdot}9 . \quad \supset \vdash : \sim(\sim p) . \supset . p$ (4)

$\vdash . (2) . (3) . (4) .$ Syll. $\supset \vdash .$ Prop.

The above is the first use of ★2·91. It asserts that a proposition must be true if it can be deduced from the supposition that it is false.

★3·42 $\vdash :. p \supset q . \supset : \sim p \supset q . \supset . q$

Dem.

$\vdash . \star 3{\cdot}22 . \quad \supset \vdash :. p \supset q . \supset : \sim q \supset \sim p$ (1)

$\vdash .$ Syll $\dfrac{\sim q , \sim p , q}{p , q , r} . \supset \vdash :. \sim q \supset \sim p . \supset : \sim p \supset q . \supset . \sim q \supset q$ (2)

$\vdash . \star 3{\cdot}12 \dfrac{\sim p \supset q , \sim q \supset q , q}{p , \quad q , \quad r} . \supset$

$\vdash :: \sim q \supset q . \supset . q : \supset :. \sim p \supset q . \supset . \sim q \supset q : \supset : \sim p \supset q . \supset . q$ (3)

$\vdash . (3) . \star 3{\cdot}41 . \quad \supset \vdash :. \sim p \supset q . \supset . \sim q \supset q : \supset : \sim p \supset q . \supset . q$ (4)

$\vdash . (1) . (2) . (4) .$ Syll. $\supset \vdash .$ Prop.

·43 $\vdash :. \sim p \supset q . \supset : p \supset q . \supset . q$ [★3·42 . Comm.]

★3·44 $\vdash :. p \supset q . \supset : p \supset \sim q . \supset . \sim p$

Dem.

$\vdash . \star 3{\cdot}22 . \quad \supset \vdash :. p \supset q . \supset : \sim q \supset \sim p$ (1)

$\vdash . \star 3{\cdot}12 \dfrac{\sim q , \sim p}{q , r} . \supset \vdash :. \sim q \supset \sim p . \supset : p \supset \sim q . \supset . p \supset \sim p$ (2)

$\vdash . \star 3{\cdot}12 \dfrac{p \supset \sim q ; p \supset \sim p , \sim p}{p , \quad q , \quad r} . \supset$

$\vdash :: p \supset \sim p . \supset . \sim p : \supset :. p \supset \sim q . \supset . p \supset \sim p : \supset : p \supset \sim q . \supset . \sim p$ (3)

$\vdash . (3) . \star 2{\cdot}91 . \quad \supset \vdash :. p \supset \sim q . \supset . p \supset \sim p : \supset : p \supset \sim q . \supset . \sim p$ (4)

$\vdash . (1) . (2) . (4) .$ Syll. $\supset \vdash .$ Prop.

·45 $\vdash :. p \supset q . \supset . q : \supset . \sim p \supset q$

Dem.

$\vdash .$ Syll $\dfrac{\sim p , p \supset q , q}{p , \quad q , \quad r} . \supset \vdash :: \sim p . \supset . p \supset q : \supset :. p \supset q . \supset . q : \supset . \sim p \supset q$ (1)

$\vdash . (1) . \star 3{\cdot}31 . \quad \supset \vdash .$ Prop.

$\star 3{\cdot}46 \quad \vdash :. \sim p \supset q . \supset . q : \supset . p \supset q \quad \left[\star 3{\cdot}45 \frac{\sim p}{p}\right]$

Note. In all formulæ concerned only with implications, p may be substituted for $\sim(\sim p)$, in virtue of $\star 2{\cdot}9$ and $\star 3{\cdot}2$. In future, this substitution will be made (as it has been in proving $\star 3{\cdot}46$) tacitly; it would be justified, in each case, by exactly similar steps, and it is not worth while to repeat them explicitly on each occasion.

$\cdot 47 \quad \vdash :. p \supset q . \supset . q : \supset : q \supset p . \supset . p$

Dem.

$\vdash . \star 3{\cdot}45 . \quad \supset \vdash :. p \supset q . \supset . q : \supset . \sim p \supset q$ (1)

$\vdash . \star 3{\cdot}21 . \quad \supset \vdash : \sim p \supset q . \supset . \sim q \supset p$ (2)

$\vdash . \star 3{\cdot}43 \frac{q , p}{p , q} . \quad \supset \vdash :. \sim q \supset p . \supset : q \supset p . \supset . p$ (3)

$\vdash . (1) . (2) . (3) . \text{Syll.} \quad \supset \vdash . \text{Prop.}$

$\star 3{\cdot}5 \quad \vdash :. p . \supset . p \supset q : \supset . p \supset q$

Dem.

$\vdash . \star 3{\cdot}43 \frac{p \supset q}{q} . \supset \vdash :: \sim p . \supset . p \supset q : \supset :. p . \supset . p \supset q : \supset . p \supset q$ (1)

$\vdash . (1) . \star 3{\cdot}31 . \quad \supset \vdash . \text{Prop.}$

$\cdot 51 \quad \vdash :. p \supset q . \supset . p : \supset . p \quad \left[\star 3{\cdot}5 \cdot \star 3{\cdot}47 \frac{p \supset q}{q}\right]$

$\star 3{\cdot}6 \quad \vdash :. p . \supset . q \supset r : \supset : p \supset q . \supset . p \supset r$

Dem.

$\vdash . \text{Comm.} \quad \supset \vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$ (1)

$\vdash . \star 3{\cdot}12 \frac{p \supset r}{r} . \supset \vdash :: q . \supset . p \supset r : \supset :. p \supset q . \supset : p . \supset . p \supset r$ (2)

$\vdash . \star 3{\cdot}12 \frac{p \supset q , p . \supset . p \supset r , p \supset r}{p , \quad q , \quad r} . \supset$

$\vdash ::. p . \supset . p \supset r : \supset . p \supset r :. \supset :: p \supset q . \supset : p . \supset . p \supset r :.$

$\supset :. p \supset q . \supset . p \supset r$ (3)

$\vdash . 3{\cdot}5 \frac{r}{q} . \supset \vdash :. p . \supset . p \supset r : \supset . p \supset r$ (4)

$\vdash . (3) . (4) . \supset \vdash :: p \supset q . \supset : p . \supset . p \supset r :. \supset :. p \supset q . \supset . p \supset r$ (5)

$\vdash . (1) . (2) . (5) . \text{Syll.} \quad \supset \vdash . \text{Prop.}$

★3·61 $\vdash :: p . \supset . q \supset r : \supset :. p . \supset . r \supset s : \supset : p . \supset . q \supset s$

Dem.

$\vdash . ★3{\cdot}12 \frac{q \supset r,\ r \supset s . \supset . q \supset s}{q, \qquad r} . \supset$

$\vdash ::. q \supset r . \supset : r \supset s . \supset . q \supset s :. \supset :: p . \supset . q \supset r : \supset :. p .$

$\supset : r \supset s . \supset . q \supset s$ (1)

$\vdash . (1) . ★3{\cdot}12 \frac{q, r, s}{p, q, r} . \supset \vdash :: p . \supset . q \supset r : \supset :. p . \supset : r \supset s . \supset . q \supset s$ (2)

$\vdash . ★3{\cdot}6 \frac{r \supset s, q \supset s}{q, r} . \quad \supset \vdash :: p . \supset : r \supset s . \supset . q \supset s :. \supset :. p .$

$\supset . r \supset s : \supset : p . \supset . q \supset s$ (3)

$\vdash . (2) . (3) .$ Syll. $\quad \supset \vdash .$ Prop.

★3·62 $\vdash :. p \supset q . \supset . p \supset r : \supset : p . \supset . q \supset r$

Dem.

$\vdash .$ Simp. $\supset \vdash : q . \supset . p \supset q$ (1)

$\vdash . ★3{\cdot}1 \frac{p \supset q, p \supset r}{p, q} . \supset \vdash :: p \supset q . \supset :. p \supset q . \supset . p \supset r : \supset . p \supset r$ (2)

$\vdash . (1) . (2) .$ Syll. $\supset \vdash :: q . \supset :. p \supset q . \supset . p \supset r : \supset . p \supset r$ (3)

$\vdash .$ Comm $\frac{q, p \supset q . \supset . p \supset r, p \supset r}{p, \qquad q, \qquad r} . \supset$

$\vdash ::. q . \supset :. p \supset q . \supset . p \supset r : \supset . p \supset r :: \supset :: p \supset q . \supset . p \supset r :$

$\supset : q . \supset . p \supset r$ (4)

$\vdash . (3) . (4) . \supset \vdash .$ Prop.

★4. *Multiplication and Addition.*

In this section we shall be concerned with the fundamental properties of the *propositional product* and the *propositional sum* of two entities p and q. The *propositional product* is practically "p and q are both true". But this, as it stands, would have to be a new primitive idea, and it is desirable to avoid primitive ideas whenever we can. Now we have seen (★3·38, note) that $\sim (p \supset \sim q)$ is true when and only when p and q are both true. Hence we may take $\sim (p \supset \sim q)$ as the *propositional product* of p and q, since this implies, or is implied by, anything that "p and q are both true" implies, or is implied by.

The *propositional sum* of p and q is practically "either p is true or q is true". We avoid a new primitive idea by taking as the propositional sum $\sim p \supset q$, *i. e.*, "if p is not true, then q is true", which is obviously equivalent to "either p or q is true". There are some advantages in taking the propositional sum as a primitive idea instead of "$p \supset q$"; we then define "$p \supset q$" as the propositional sum of $\sim p + q$. The choice is a matter of taste.

Both the propositional product and the propositional sum of p and q are propositional functions of p and q. They are *significant* for all values of p and q, *i. e.*, even for values which are not propositions; but they are not *important* unless p and q are propositions.

In a definition, we write the term, or combination of terms, to be defined to the left of the sign of equality, and the defining combination of terms to the right; at the end we write "Df." The sign of equality and the "Df." are to be considered as together forming one symbol, meaning "is defined to mean". A definition is not properly part of the subject, being concerned with the symbols, not with what they symbolize; nevertheless, *practically*, the definitions are usually the most important part in a study of principles.

The sign of equality *not* followed by "Df." will be used to express identity, and must not be confounded with the sign of equality followed by "Df."

★4·1 $p \,.\, q \,. \quad = .\, \sim(p \supset \sim q)$ Df.

·11 $p \vee q \,. \quad = .\, \sim p \supset q$ Df.

·12 $p \equiv q \,. \quad = .\, p \supset q \,.\, q \supset p$ Df.

·13 $p \supset q \supset r \,. = .\, p \supset q \,.\, q \supset r$ Df.

·14 $p \equiv q \equiv r \,. = .\, p \equiv q \,.\, q \equiv r$ Df.

Definitions have no assertion-signs, because they are not expressions of propositions, but of volitions. Of the above definitions, ★4·1·11 have been already explained. ★4·13·14 are merely convenient abbreviations. ★4·12 defines *equivalence*: "$p \equiv q$" is read "p is equivalent to q". This holds when and only when p and q are either both true or both not true. In the theory of implication it plays a part analogous to that of equality in ordinary mathematics.

When it is necessary, the propositional product of two terms will be expressed by placing several dots between them. Thus, *e. g.*,

$$p \supset q \,.\supset .\, q : q \supset p$$

will express the propositional product of $p \supset q \,.\supset .\, q$ and $q \supset p$;

$$p\,.\supset:q\supset r\,.\supset.\,s:.\,q\,.\,s\,.\supset:p:p\supset q\,.\supset.\,r$$

will express the propositional product of

$$p\,.\supset:q\supset r\,.\supset.\,s \text{ and } q\,.\,s\,.\supset:p:p\supset q\,.\supset.\,r$$

where $q\,.\,s\,.\supset:p:p\supset q\,.\supset.\,r$ expresses that the propositional product of q and s implies the propositional product of p and $p\supset q\,.\supset.\,r$.

★4·2 $\vdash:.\,p\,.\supset:q\,.\supset.\,p\,.\,q$ [★3·38]

·21 $\vdash:.\,q\,.\supset:p\,.\supset.\,p\,.\,q$ [★4·2. Comm.]

·22 $\vdash:p\,.\,q\,.\supset.\,q\,.\,p$

Dem.

$\vdash.\,$★$2{\cdot}92\dfrac{q,p}{p,q}.\supset\vdash:q\supset\sim p\,.\supset.\,p\supset\sim q$ (1)

$\vdash.\,$★$3{\cdot}22\dfrac{q\supset\sim p,\,p\supset\sim q}{p,q}.\supset$

$\vdash:.\,q\supset\sim p\,.\supset.\,p\supset\sim q:\supset:\sim(p\supset\sim q)\,.\supset.\sim(q\supset\sim p)$ (2)

$\vdash.\,(1)\,.\,(2)\,.\supset\vdash.$ Prop.

·23 $\vdash:p\lor q\,.\supset.\,q\lor p$ [★3·21]

·24 $\vdash.\sim(p\,.\sim p)$

Dem.

$\vdash.\,$★$3{\cdot}2.\quad\supset\vdash.\,p\supset\sim(\sim p).$ (1)

$\vdash.\,$★$3{\cdot}2.\quad\supset\vdash:p\supset\sim(\sim p)\,.\supset.\sim[\sim\{p\supset\sim(\sim p)\}]$ (2)

$\vdash.\,$Id. (★4·1). $\supset\vdash:\sim[\sim\{p\supset\sim(\sim p)\}]\,.\supset.\sim(p\,.\sim p)$ (3)

$\vdash.\,(2)\,.\,(3)\,.$ Syll. $\supset\vdash:p\supset\sim(\sim p)\,.\supset.\sim(p\,.\sim p)$ (4)

$\vdash.\,(1)\,.\,(4).\quad\supset\vdash$ Prop.

The above is the law of contradiction, in the form "nothing is both true and not true". When a definition is referred to, as in the third line of the above proof, it is adduced in round brackets, because it is not *logically* relevant, but only *symbolically*.

Proofs may be made both shorter and easier by the following plan. Suppose we know some proposition $(C\between p,q)$, and suppose that some previously proved proposition (a) gives

$$(C\between p,q)\,.\supset.\,(D\between p,q).$$

We may then write the proof of $(D\between p,q)$ in the form

$$\vdash.\,(C\between p,q)\,.$$

[(a)] $\quad\supset\vdash.\,(D\between p,q),$

where the proposition by which the transition is made is indicated in square brackets on the left. Thus in ★4·24 we should write

$$\begin{array}{ll} \vdash . \star 3\cdot 2 . & \supset \vdash . p \supset \sim(\sim p) . \\ {[}\star 3\cdot 2] & \supset \vdash . \sim [\sim \{p \supset \sim(\sim p)\}] \\ {[}\text{Id.}\,(\star 4\cdot 1)] & \supset \vdash . \sim (p . \sim p) \end{array}$$

Another similar abbreviation is got by the use of ★4·13. Suppose we have

$$\begin{array}{ll} \vdash . (a) . & \supset \vdash : (A \between p , q) . \supset . (B \between p , q) \\ \vdash . (b) . & \supset \vdash : (B \between p , q) . \supset . (C \between p , q) \\ \vdash . (c) . & \supset \vdash : (C \between p , q) . \supset . (D \between p , q) \end{array}$$

and we wish to prove $(A \between p , q) . \supset . (D \between p , q)$. We write

$$\begin{array}{lr} \vdash . (a) . \ \supset \vdash : (A \between p , q) . \supset . (B \between p , q) . & \\ {[}(b)] \qquad\qquad\qquad\qquad\quad \supset . (C \between p , q) . & \\ {[}(c)] \qquad\qquad\qquad\qquad\quad \supset . (D \between p , q) & \qquad\qquad (1) \end{array}$$

where "(1)" means "$(A \between p , q) . \supset . (D \between p , q)$", which follows from the above by means of *Syll.* This abbreviation is of very great convenience.

★4·25 $\vdash . p \vee \sim p \quad \left[\text{Id}\ \dfrac{\sim p}{p}\right]$

This is the law of excluded middle, in the form "everything is either true or not true".

·26 $\vdash : p . q . \supset . p \qquad [\star 3\cdot 36]$

·27 $\vdash : p . q . \supset . q \qquad [\star 3\cdot 37]$

·28 $\vdash : p . \supset . p \vee q \qquad [\star 3\cdot 3]$

·29 $\vdash : q . \supset . p \vee q \qquad \left[\text{Simp}\ \dfrac{q , \sim p}{p , q}\right]$

★4·3 $\vdash :. p . q . \supset . r : \supset : p . \supset . q \supset r \qquad$ ["Exp"]

This is an important principle of inference, called by Peano "exportation", because the q is exported out of the hypothesis. It will be adduced as "Exp".

Dem.

$$\begin{array}{ll} \vdash . \text{Id.}\,(\star 4\cdot 1) . \supset \vdash :. p . q . \supset . r : \supset : \sim (p \supset \sim q) . \supset . r : & \\ {[}\star 3\cdot 21] & \supset : \sim r . \supset . p \supset \sim q : \\ {[}\text{Comm}] & \supset : p . \supset . \sim r \supset \sim q : \\ {[}\star 3\cdot 23\cdot 12] & \supset : p . \supset . q \supset r :. \supset \vdash . \text{Prop.} \end{array}$$

In this proof we use the second of the abbreviations mentioned in ★4·24 *note*. The fact that there are *two* dots, not *three*, at the ends of the lines (except the last line), indicates that it is not the whole of the first line, but only its consequent, that implies the second line, and so with the transition from the second line to the third and from the third to the fourth. Thus, by *Syll*, the hypothesis of the first line implies each successive subsequent line. The three dots before "$\supset \vdash . \text{Prop.}$" indicate that now the hypothesis of the first line is again to be taken into account.

★4·31 $\vdash :. p . \supset . q \supset r : \supset : p . q . \supset . r$ ["Imp"]

This principle is of equal importance with ★4·3, and is called by Peano "importation", because q is imported into the hypothesis. It will be adduced as "Imp".

Dem.

$\vdash . \star 3\cdot22\cdot12 \cdot \supset \vdash :. p . \supset . q \supset r : \supset : p . \supset . \sim r \supset \sim q :$

[Comm] $\supset : \sim r \supset . p \supset \sim q :$

[★3·21] $\supset : \sim (p \supset \sim q) . \supset . r :$

[(★4·1)] $\supset : p . q . \supset . r :. \supset \vdash . \text{Prop.}$

★4·32 $\vdash :. p . q . \supset . r : \equiv : p . \supset . q \supset r : \equiv : q . \supset . p \supset r : \equiv : q . p . \supset . r$

Dem.

$\vdash . \star 4 \cdot 2 \dfrac{\text{Exp}, \text{Imp}}{p , q} . \quad \supset \vdash :. \text{Exp} . \supset : \text{Imp} . \supset . \text{Exp} . \text{Imp}.$ (1)

$\vdash . (1) . \text{Exp}. \quad \supset \vdash : \text{Imp} . \supset . \text{Exp} . \text{Imp}.$ (2)

$\vdash . (2) . \text{Imp}. \quad \supset \vdash : \text{Exp} . \text{Imp} . :$

[Id. (★4·12)] $\supset \vdash :. p . q . \supset . r : \equiv : p . \supset . q \supset r$ (3)

$\vdash . \text{Comm}. \quad \supset \vdash :. p . \supset . q \supset r : \supset : q . \supset . p \supset r$ (4)

$\vdash . \text{Comm} \dfrac{q , p}{p ; q} . \quad \supset \vdash :. q . \supset . p \supset r : \supset : p . \supset . q \supset r$ (5)

$\vdash . \{(4) . (5)\} . \star 4 \cdot 2 . \quad \supset \vdash :. p . \supset . q \supset r : \equiv : q . \supset . p \supset r$ (6)

$\vdash . \{\text{Imp} . \text{Exp} .\} . \star 4 \cdot 2 . \supset \vdash :. q . \supset . p \supset r : \equiv : q . p . \supset . r$ (7)

$\vdash . \{(3) . (6)\} . (7) . \quad \supset \vdash . \text{Prop.}$

Note. The use of ★4·2, which is given at length above in proving $\vdash . \text{Exp. Imp.}$, will in future be assumed, as it is above in passing from (4) and (5) to (6), and from *Imp* and *Exp* to (7). By ★4·2, when two propositions have been proved, it follows that their propositional product is true; but it would be unnecessarily lengthy to go through the steps each time this principle is used. Observe that (1), in the above proof, is obtained from ★4·2 by the use of ★2·2, not of ★2·3.

★4·33 $\vdash : p \supset q . q \supset r . \supset . p \supset r$ [Syll. Imp.]

·34 $\vdash : q \supset r . p \supset q . \supset . p \supset r$ [★3·12 . Imp.]

These two propositions will hereafter be referred to as "Syll"; they are usually more convenient than either ★2·7 or ★3·12.

·35 $\vdash : p . p \supset q . \supset . q$ ["Ass"]

[★3·1 . Imp.]

This is an important principle of inference, which I shall call the "principle of assertion", and refer to as "Ass".

·36 $\vdash : \sim q . p \supset q . \supset . \sim p$ [★3·22 . Comm. Imp.]

·37 $\vdash :. p . q . \supset . r : \supset : p . \sim r . \supset . \sim q$

Dem.

$\vdash . ★3{\cdot}22 . \supset \vdash : q \supset r . \supset . \sim r \supset \sim q :$

[★3·12] $\supset \vdash :. p . \supset . q \supset r : \supset : p . \supset . \sim r \supset \sim q$ (1)

$\vdash . \text{Exp.}$ $\supset \vdash :. p . q . \supset . r : \supset : p . \supset . q \supset r$ (2)

$\vdash . \text{Imp.}$ $\supset \vdash :. p . \supset . \sim r \supset \sim q : \supset : p . \sim r . \supset . \sim q$ (3)

$\vdash . (2) . (1) . (3) . \text{Syll.} \supset \vdash . \text{Prop.}$

This is another form of transposition, and is a principle constantly employed.

★4·4 $\vdash : p . q . \supset . p \supset q$ $\left[★3{\cdot}34 \frac{\sim q}{q} \right]$

·41 $\vdash :. p \supset r . \supset : p . q . \supset . r$ [★4·26 . Syll.]

·42 $\vdash :. q \supset r . \supset : p . q . \supset . r$ [★4·27 . Syll.]

·43 $\vdash :. p \supset q . p \supset r . \supset : p . \supset . q . r$ ["Comp"]

This is called by Peano the "principle of composition"; it asserts that if p implies each of two propositions, it implies their propositional product. It will be referred to as "Comp".

Dem.

$\vdash . \star 4\cdot 2 . \supset \vdash :: q . \supset :. r . \supset . q . r :.$

$[\star 3\cdot 12] \qquad \supset :. p \supset r . \supset : p . \supset . q . r :.$

$[\text{Comm}] \qquad \supset :. p . \supset : p \supset r . \supset . q . r \qquad (1)$

$\vdash . (1) . \star 3\cdot 12 . \supset \vdash ::. p \supset q . \supset :: p . \supset :. p . \supset : p \supset r . \supset . q . r ::$

$[\star 3\cdot 5] \qquad \supset :: p . \supset : p \supset r . \supset . q . r ::$

$[\text{Comm}] \qquad \supset :: p \supset r . \supset : p . \supset . q . r \qquad (2)$

$\vdash . (2) . \text{Imp}. \quad \supset \vdash . \text{Prop}.$

4·44 $\vdash :. q \supset p . r \supset p . \supset : q \vee r . \supset . p$ ["Alt."]

This principle is analogous to $\star 4\cdot 43$; I shall call it the "principle of the alternative" and refer to it as "Alt". The analogy between $\star 4\cdot 43$ and $\star 4\cdot 44$ is of a sort which generally subsists between formulæ concerning products and formulæ concerning sums.

Dem.

$\vdash . \text{Syll}. \supset \vdash :. \sim q \supset r . r \supset p . \supset : \sim q \supset p :$

$[\star 3\cdot 43] \qquad \supset : q \supset p . \supset . p \qquad (1)$

$\vdash . (1) . \text{Exp}. \supset \vdash :: \sim q \supset r . \supset :. r \supset p . \supset : q \supset p . \supset . p :.$

$[\text{Comm. Imp}] \qquad \supset :. q \supset p . r \supset p . \supset . p \qquad (2)$

$\vdash . (2) . \text{Comm}. \quad \supset \vdash :. q \supset p . r \supset p . \supset : \sim q \supset r . \supset . p :.$

$[(\star 4\cdot 11)] \quad \supset \vdash . \text{Prop}.$

4·45 $\vdash :. p \supset q . \supset : p . r . \supset . q . r$ ["Fact."]

This principle shows that we may multiply both sides of an implication by a common factor; hence it is called by Peano the "principle of the factor". I shall refer to it as "Fact".

Dem.

$\vdash . \text{Syll} \frac{\sim r}{r} . \supset \vdash :. p \supset q . \supset : q \supset \sim r . \supset . p \supset \sim r :$

$[\star 3\cdot 22] \qquad \supset : \sim (p \supset \sim r) . \supset . \sim (q \supset \sim r) :.$

$[(\star 4\cdot 1)] \quad \supset \vdash . \text{Prop}.$

★4·46 $\vdash :. p \supset q . \supset : p \vee r . \supset . q \vee r$ ["Sum."]

This principle is analogous to ★4·45; it may be called the "principle of the summand". I shall refer to it as "Sum".

Dem.

$\vdash . \star 3\cdot22 . \supset \vdash :. p \supset q . \supset : \sim q \supset \sim p :$

[Syll] $\supset : \sim p \supset r . \supset . \sim q \supset r :.$

[(★4·11)] $\supset \vdash .$ Prop.

★4·47 $\vdash :. p \supset r . q \supset s . \supset : p . q . \supset . r . s$

This proposition, or rather its analogue for classes, was proved by Leibniz, and evidently pleased him, since he calls it "præclarum theorema".*

Dem.

$\vdash . \star 4\cdot26 . \supset \vdash :. p \supset r . q \supset s . \supset : p \supset r :$

[Fact] $\supset : p . q . \supset . r . q :$

[★4·22] $\supset : p . q . \supset . q . r$ (1)

$\vdash . \star 4\cdot27 . \supset \vdash :. p \supset r . q \supset s . \supset : q \supset s :$

[Fact] $\supset : q . r . \supset . s . r :$

[★4·22] $\supset : q . r . \supset . r . s$ (2)

$\vdash . (1) . (2) . \star 4\cdot2 . \supset \vdash :: p \supset r . q \supset s . \supset : p . q . \supset . q . r :. p \supset r . q \supset s .$
$\supset : q . r . \supset . r . s ::$

[Comp] $\supset \vdash :: p \supset r . q \supset s . \supset :. p . q . \supset . q . r : q . r . \supset . r . s :.$

[Syll] $\supset :. p . q . \supset . r . s :: \supset \vdash .$ Prop.

★4·48 $\vdash :. p \supset r . q \supset s . \supset : p \vee q . \supset . r \vee s$

This theorem is the analogue of ★4·47.

Dem.

$\vdash . \star 4\cdot26 . \supset \vdash :. p \supset r . q \supset s . \supset : p \supset r :$

[Sum] $\supset : p \vee q . \supset . r \vee q :$

[★4·23] $\supset : p \vee q . \supset . q \vee r$ (1)

$\vdash . \star 4\cdot27 . \supset \vdash :. p \supset r . q \supset s . \supset : q \supset s :$

[Sum] $\supset : q \vee r . \supset . s \vee r :$

[★4·23] $\supset : q \vee r . \supset . r \vee s$ (2)

$\vdash . (1) . (2) . \star 4\cdot2 . \mathrm{Comp} . \supset$
$\vdash :. p \supset r . q \supset s . \supset : p \vee q . \supset . q \vee r : q \vee r . \supset . r \vee s :$

[Syll] $\supset : p \vee q . \supset . r \vee s :. \supset \vdash .$ Prop.

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*Philosophical works, Gerhardt's edition, Vol. VII, p. 223.

★4·5 $\vdash :. p \vee q . \equiv : p \supset q . \supset . q$ [★3·43·45]

·51 $\vdash :. p \supset q . \supset . q : \equiv : q \supset p . \supset . p$ $\left[★3·47. \quad ★3·47 \dfrac{q, p}{p, q} \right]$

·6 $\vdash :. p \equiv q . \supset : p \supset r . \equiv . q \supset r$

Dem.

$\vdash . ★4·22 . \supset \vdash : p \equiv q . \supset . q \supset p . p \supset q$ (1)

$\vdash .$ Syll. $\supset \vdash :. q \supset p . \supset : p \supset r . \supset . q \supset r$ (2)

$\vdash .$ Syll. $\supset \vdash :. p \supset q . \supset : q \supset r . \supset . p \supset r$ (3)

$\vdash . (2) . (3) . ★4·47 . \supset$

$\vdash :. q \supset p . p \supset q . \supset : p \supset r . \equiv . q \supset r$ (4)

$\vdash . (1) . (4) .$ Syll. $\supset \vdash .$ Prop.

·61 $\vdash :. p \equiv q . \supset : r \supset p . \equiv . r \supset q$ [Similar proof.]

★5 *Formal rules.*

In this section, we shall be concerned with rules analogous, more or less, to those of ordinary algebra. It is from these rules that the usual "calculus of formal logic" starts. Treated as a "calculus", the rules of deduction are capable of many other interpretations. But all other interpretations depend upon the one here considered, since in all of them we deduce consequences from our rules, and thus presuppose the theory of deduction. One very simple interpretation of the "calculus" is as follows: The entities considered are to be numbers which are all either 0 or 1; "$p \supset q$" is to have the value 0 if p is 1 and q is 0; otherwise it is to have the value 1; $\sim p$ is to be 1 if p is 0, and 0 if p is 1; $p . q$ is to be 1 if p and q are both 1, and is to be 0 in any other case; $p \vee q$ is to be 0 if p and q are both 0, and is to be 1 in any other case; and the assertion-sign is to mean that what follows has the value 1.

To show that our primitive propositions are sufficient for the propositional calculus, we may compare our propositions with the first set of postulates in Huntington's "Sets of Independent Postulates for the Algebra of Logic".* In making this comparison, observe that his class K is replaced by the class of all the entities, so that postulates concerning membership of K are necessarily satisfied by everything. His two rules of combination are our addition and multiplication; his $\wedge$ is our $(s) . s$ [see ★7], and his V is our $\sim (s) . s$. Then his postulates I*a* and I*b* are satisfied because the class K is the class of all entities; his equality is replaced by our equivalence, so that II*a* is our ★7·3, and II*b* is ★7·31, III*a* and III*b* are respectively ★5·31 and ★5·3. IV*a* is ★5·41, and IV*b* is ★5·4; V results from ★7·34·35, and VI results from ★7·26. Thus our material (together with the additions to be made in ★7) is sufficient to found the propositional calculus. Symbolic logic considered as a calculus has undoubtedly much interest on its own account; but in my opinion this aspect has hitherto been too much emphasized, at the expense of the aspect in which symbolic logic is merely the most elementary part of mathematics, and the logical prerequisite of all the rest. For this reason, I shall only deal briefly with what is required for the algebra of symbolic logic.

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*Trans. Amer. Math. Soc., July, 1904, p. 292.

★5·1 $\vdash : p \supset q . \equiv . \sim q \supset \sim p$ [★3·22·23]

·11 $\vdash : p \equiv q . \equiv . \sim p \equiv \sim q$ [★3·22·23. ★4·47·22]

·12 $\vdash : p \equiv \sim q . \equiv . q \equiv \sim p$ [★2·92. ★3·21]

·13 $\vdash . p \equiv \sim (\sim p)$ [★2·9. ★3·2]

·14 $\vdash :. p . q . \supset . r : \equiv : p . \sim r . \supset . \sim q$ [★4·37. ★5·13]

Observe that if $p \equiv q$, q may be substituted for p, or vice versa, in any formula involving no primitive ideas except implication and negation, without altering the truth or falsehood of the formula. This can be proved for each separate case, but not generally, because we have no means of specifying (with our apparatus of primitive ideas) that a complex $(C \between p, q)$ is to be one that can be built up out of implication and negation alone.

·15 $\vdash :. p . q . \supset . \sim r : \equiv : q . r . \supset . \sim p$ [★4·22. ★5·13·14]

★5·2 $\vdash . p \equiv p$ [Id. ★4·2]

·21 $\vdash : p \equiv q . \equiv . q \equiv p$ [★4·22]

·22 $\vdash : p \equiv q . q \equiv r . \supset . p \equiv r$

Dem.

$\vdash .$ ★4·26. $\supset \vdash : p \equiv q . q \equiv r . \supset . p \equiv q .$
[★4·26] $\supset . p \supset q$ (1)
$\vdash .$ ★4·27. $\supset \vdash : p \equiv q . q \equiv r . \supset . q \equiv r .$
[★4·26] $\supset . q \supset r$ (2)
$\vdash . (1) . (2) .$ Comp. $\supset \vdash : p \equiv q . q \equiv r . \supset . p \supset q . q \supset r .$
[Syll] $\supset . p \supset r$ (3)

$\vdash$. ★4.27. $\supset \vdash : p \equiv q . q \equiv r . \supset . q \equiv r .$
[★4·27] $\supset . r \supset q$ (4)
$\vdash$. ★4·26. $\supset \vdash : p \equiv q . q \equiv r . \supset . p \equiv q .$
[★4·27] $\supset . q \supset p$ (5)
$\vdash$. (4) . (5) . Comp. $\supset \vdash : p \equiv q . q \equiv r . \supset . r \supset q . q \supset p .$
[Syll] $\supset . r \supset p$ (6)
$\vdash$. (3) . (6) . Comp. $\supset \vdash$. Prop.

Note. The above three propositions show that the relation of equivalence is reflexive (★5·2), symmetrical (★5·21), and transitive (★5·22). Implication is reflexive and transitive, but not symmetrical. The properties of being symmetrical, transitive, and (at least within a certain field) reflexive are essential to any relation which is to have the formal characters of equality.

★5·24 $\vdash : p . \equiv . p . p$
Dem.
$\vdash$. ★4·26 . $\supset \vdash : p . p . \supset . p$ (1)
$\vdash$. ★4·2 . $\supset \vdash :. p . \supset : p . \supset . p . p . :$
[★3·5] $\supset \vdash : p . \supset . p . p$ (2)
$\vdash$. (1) . (2) . ★4·21. $\supset \vdash$. Prop.

·25 $\vdash : p . \equiv . p \lor p$
Dem.
$\vdash$. ★4·28. $\supset \vdash : p . \supset . p \lor p$ (1)
$\vdash$. Id. (★4·11). $\supset \vdash : p \lor p . \supset . \sim p \supset p .$
[★3·41] $\supset . p$ (2)
$\vdash$. (1) . (2). ★4·2. $\supset \vdash$. Prop.

Note. ★5·24·25 are the two forms of the *law of tautology*, which is what chiefly distinguishes the algebra of symbolic logic from ordinary algebra.

★5·3 $\vdash : p . q . \equiv . q . p$ [★4·22]

Note. Whenever we have, whatever values p and q may have

$$(C \between p, q) . \supset . (C \between q, p),$$

we have also

$$(C \between p, q) . \equiv . (C \between q, p).$$

For $\{(C\between p,q).\supset.(C\between q,p)\}\frac{q,p}{p,q}.\supset:(C\between q,p).\supset.(C\between p,q).$

·31 $\vdash:p\vee q.\equiv.q\vee p$ [★3·21]

★5·32 $\vdash:(p.q).r.\equiv.p.(q.r)$

Dem.

$\vdash.$★5·15. $\supset\vdash:.p.q.\supset.\sim r:\equiv:q.r.\supset.\sim p:$

[★5·12] $\equiv:p.\supset.\sim(q.r)$ (1)

$\vdash.(1).$★5·11. $\supset\vdash:\sim(p.q.\supset.\sim r).\equiv.\sim(p.\supset.\sim(q.r)):$

[(★4·1)] $\supset\vdash.$ Prop.

Note. Here "(1)" stands for "$\vdash:.p.q.\supset.\sim r:\equiv:p.\supset.\sim(q.r)$", which is obtained from the above steps by ★5·22. The use of ★5·22 will often be tacit, as above. The principle is the same as that explained in respect of implication in ★4·24 *note.*

·33 $\vdash:(p\vee q)\vee r.\equiv.p\vee(q\vee r)$

Dem.

$\vdash.$★5·2.(★4·11)$.\supset\vdash:.(p\vee q)\vee r.\equiv:\sim(\sim p\supset q).\supset.r:$

[★5·1] $\equiv:\sim r.\supset.\sim p\supset q:$

[★4·32] $\equiv:\sim p.\supset.\sim r\supset q:$

[★5·1·★4·61] $\equiv:\sim p.\supset.\sim q\supset r:$

[★5·2·(★4·11)] $\equiv:p\vee(q\vee r):.\supset\vdash.$ Prop.

The above are the associative laws for multiplication and addition. To avoid brackets, we introduce the following definitions:

·34 $p.q.r.=.(p.q).r$ Df.

·35 $p\vee q\vee r.=.(p\vee q)\vee r$ Df.

★5.36 $\vdash:.p\equiv q.\supset:p.r.\equiv.q.r$ [Fact. ★4·47]

·37 $\vdash:.p\equiv q.\supset:p\vee r.\equiv.q\vee r$ [Sum. ★4·47]

·38 $\vdash:.p\equiv r.q\equiv s.\supset:p.q.\equiv.r.s$ [★4·47. ★5·32. ★4·22]

·39 $\vdash:.p\equiv r.q\equiv s.\supset:p\vee q.\equiv.r\vee s$ [★4·48·47. ★5·32. ★4·22]

$\star 5\cdot 4 \quad \vdash :. p . q \vee r . \equiv : p . q . \vee . p . r$

This is the first form of the distributive law.

Dem.

$\vdash . \star 4\cdot 2 . \quad \supset \vdash :: p . \supset :. q . \supset . p . q :. p . \supset : r . \supset . p . r ::$

$[\text{Comp.}] \quad \supset \vdash :: p . \supset :. q . \supset . p . q : r . \supset . p . r :.$

$[\star 4\cdot 48] \quad \supset :. q \vee r . \supset : p . q . \vee . p . r$ (1)

$\vdash . (1) . \text{Imp.} \quad \supset \vdash :. p . q \vee r . \supset : p . q . \vee . p . r$ (2)

$\vdash . \star 4\cdot 26 . \quad \supset \vdash :. p . q . \supset . p : p . r . \supset . p :.$

$[\text{Alt}] \quad \supset \vdash :. p . q . \vee . p . r : \supset . p$ (3)

$\vdash . \star 4\cdot 27 . \quad \supset \vdash :. p . q . \supset . q : p . r . \supset . r :.$

$[\star 4\cdot 48] \quad \supset \vdash :. p . q . \vee . p . r : \supset . q \vee r$ (4)

$\vdash . (3) . (4) . \text{Comp.} \quad \supset \vdash :. p . q . \vee . p . r : \supset . p . q \vee r$ (5)

$\vdash . (2) . (5) . \quad \supset \vdash . \text{Prop.}$

$\star 5\cdot 41 \quad \vdash :. p . \vee . q . r : \equiv . p \vee q . p \vee r$

This is the second form of the distributive law—a form to which there is nothing analogous in ordinary algebra. By the conventions as to dots, "$p . \vee . q . r$" means "$p \vee (p . r)$".

Dem.

$\vdash . \star 4\cdot 26 . \text{Sum.} \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee q$ (1)

$\vdash . \star 4\cdot 27 . \text{Sum.} \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee r$ (2)

$\vdash . (1) . (2) . \text{Comp.} \quad \supset \vdash :. p . \vee . q . r : \supset . p \vee q . p \vee r$ (3)

$\vdash . \text{Id.} (\star 4\cdot 11) . \quad \supset \vdash :. p \vee q . p \vee r . \supset : \sim p \supset q . \sim p \supset r :$

$[\text{Comp}] \quad \supset : \sim p . \supset . q . r :$

$[\text{Id.} (\star 4\cdot 11)] \quad \supset : p . \vee . q . r$ (4)

$\vdash . (3) . (4) . \supset \vdash . \text{Prop.}$

$\cdot 42 \quad \vdash :. p . \equiv : p . q . \vee . p . \sim q$

Dem.

$\vdash . \star 4\cdot 21 . \supset \vdash :. q \vee \sim q . \supset : p . \supset . p . q \vee \sim q :.$

$[\star 4\cdot 25] \quad \supset \vdash : p . \supset . p . q \vee \sim q$ (1)

$\vdash . \star 4\cdot 26 . \supset \vdash : p . q \vee \sim q . \supset . p$ (2)

$\vdash . (1) . (2) . \supset \vdash :. p . \equiv : p . q \vee \sim q :$

$[\star 5\cdot 4] \quad \equiv : p . q . \vee . p . \sim q :. \supset \vdash . \text{Prop.}$

★5·43 $\vdash :. p . \equiv : p \vee q . p \vee \sim q$
Dem.
$\vdash . \star 4{\cdot}28 . \quad \supset \vdash : p . \supset . p \vee q : p . \supset . p \vee \sim q :$
$[\text{Comp}] \quad \supset \vdash : p . \supset . p \vee q . p \vee \sim q$ (1)
$\vdash . \star 3{\cdot}44 \frac{\sim p}{p} . \supset \vdash :. \sim p \supset q . \supset : \sim p \supset \sim q . \supset . p :.$
$[\text{Imp}] \quad \supset \vdash :. \sim p \supset q . \sim p \supset \sim q . \supset . p :.$
$[(\star 4{\cdot}11)] \quad \supset \vdash :. p \vee q . p \vee \sim q . \supset . p$ (2)
$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop.}$

·44 $\vdash :. p . \equiv : p . \vee . p . q$
Dem.
$\vdash . \star 4{\cdot}28 . \quad \supset \vdash :. p . \supset : p . \vee . p . q$ (1)
$\vdash . \text{Id.} \star 4{\cdot}26 . \supset \vdash :. p \supset p : p . q . \supset . p :.$
$[\star 4{\cdot}44] \quad \supset \vdash :. p . \vee . p . q : \supset . p$ (2)
$\vdash . (1) . (2) . \quad \supset \vdash . \text{Prop.}$

·45 $\vdash : p . \equiv . p . p \vee q \quad [\star 4{\cdot}26{\cdot}28]$

The following propositions are important. They show how to transform implications into sums or into denials of products, and vice versa. Compare ★1·2.

★5·5 $\vdash : \sim (p . q) . \equiv . p \supset \sim q \quad [\star 5{\cdot}12 . (\star 4{\cdot}1)]$
·51 $\vdash : \sim p \vee \sim q . \equiv . p \supset \sim q \quad [\star 5{\cdot}13 . (\star 4{\cdot}11)]$
·52 $\vdash : \sim (\sim p . \sim q) . \equiv . \sim p \supset q \quad [\star 5{\cdot}13]$
·53 $\vdash : p \vee q . \equiv . \sim p \supset q \quad [\star 5{\cdot}2]$
·54 $\vdash : \sim (p . \sim q) . \equiv . p \supset q \quad [\star 5{\cdot}5{\cdot}13]$
·55 $\vdash : \sim p \vee q . \equiv . p \supset q \quad [\star 5{\cdot}13]$
·56 $\vdash : \sim (\sim p . q) . \equiv . \sim p \supset \sim q \quad \left[\star 5{\cdot}5 \frac{\sim p}{p}\right]$
·57 $\vdash : p \vee \sim q . \equiv . \sim p \supset \sim q \quad \left[\star 5{\cdot}53 \frac{\sim q}{q}\right]$

From the above we obtain De Morgan's formulæ, as follows:

·6 $\vdash : \sim (p . q) . \equiv . \sim p \vee \sim q \quad [\star 5{\cdot}5{\cdot}51]$
·61 $\vdash : p . q . \equiv . \sim (\sim p \vee \sim q) \quad [\star 5{\cdot}6{\cdot}12]$
·62 $\vdash : \sim (\sim p . \sim q) . \equiv . p \vee q \quad [\star 5{\cdot}52{\cdot}53]$
·63 $\vdash : \sim p . \sim q . \equiv . \sim (p \vee q) \quad [\star 5{\cdot}62{\cdot}12]$
·64 $\vdash : \sim (p . \sim q) . \equiv . \sim p \vee q \quad [\star 5{\cdot}54{\cdot}55]$
·65 $\vdash : p . \sim q . \equiv . \sim (\sim p \vee q) \quad [\star 5{\cdot}64{\cdot}12]$
·66 $\vdash : \sim (\sim p . q) . \equiv . p \vee \sim q \quad [\star 5{\cdot}56{\cdot}57]$
·67 $\vdash : \sim p . q . \equiv . \sim (p \vee \sim q) \quad [\star 5{\cdot}66{\cdot}12]$

★5·7 $\vdash:.p\supset q.\equiv:p.\supset.p.q$

Dem.

$\vdash.$★4·26.★3·12.$\supset\vdash:.p.\supset.p.q:\supset.p\supset q$ (1)

$\vdash.$Comp. $\supset\vdash:.p\supset p.p\supset q.\supset:p.\supset.p.q:.$

[Exp] $\supset\vdash::p\supset p.\supset:.p\supset q.\supset:p.\supset.p.q::$

[Id] $\supset\vdash:.p\supset q.\supset:p.\supset.p.q$ (2)

$\vdash.(1).(2).$ $\supset\vdash.$Prop.

·71 $\vdash:.p\supset q.\equiv:p.\equiv.p.q$

Dem.

$\vdash.$★4·21. $\supset\vdash::p.q.\supset.p:\supset:.p.\supset.p.q:\supset:p.\equiv.p.q::$

[★4·26] $\supset\vdash:.p.\supset.p.q:\supset:p.\equiv.p.q$ (1)

$\vdash.$★4·26. $\supset\vdash:.p.\equiv.p.q:\supset:p.\supset.p.q$ (2)

$\vdash.(1).(2).$ $\supset\vdash:.p.\supset.p.q:\equiv:p.\equiv.p.q$ (3)

$\vdash.(3).$★5·7. $\supset\vdash.$Prop.

The above proposition enables us to transform every implication into an equivalence, which is an advantage if we wish to assimilate symbolic logic as far as possible with ordinary algebra. But when symbolic logic is regarded as an instrument of proof, we need implications, and it is merely inconvenient to substitute equivalences.

★5·72 $\vdash:.p\supset q.\equiv:q.\equiv.p\lor q$

Dem.

$\vdash.$★5·1. $\supset\vdash:.p\supset q.\equiv:\sim q\supset\sim p:$

$\left[\bigstar 5\cdot71\,\dfrac{\sim q,\sim p}{p,q}\right]$ $\equiv:\sim q.\equiv.\sim q.\sim p:$

[★5·12] $\equiv:q.\equiv.\sim(\sim q.\sim p):$

[★5·62] $\equiv:q.\equiv.q\lor p:$

[★5·31] $\equiv:q.\equiv.p\lor q:.\supset\vdash.$Prop.

·73 $\vdash:.q.\supset:p.\equiv.p.q$ [Simp. ★5·71]

This proposition is very useful, since it shows that a true factor may be omitted from a product without altering its truth or falsehood, just as a true hypothesis may be omitted from an implication.

·74 $\vdash:.\sim p.\supset:q.\equiv.p\lor q$ [★3·31.★5·72]

★5·76 $\vdash :. p \supset q . p \supset r . \equiv : p . \supset . q . r$ $\left[\star 5{\cdot}41 \dfrac{\sim p}{p} . \star 5{\cdot}55\right]$

·77 $\vdash :. q \supset p . r \supset p . \equiv : q \vee r . \supset . p$ [Alt. ★4·28·29]

·78 $\vdash :. p \supset q . \vee . p \supset r : \equiv : p . \supset . q \vee r$

Dem.

$\vdash . \star 5{\cdot}55{\cdot}39 . \supset \vdash :. p \supset q . \vee . p \supset r : \equiv : \sim p \vee q . \vee . \sim p \vee r :$

[★5·33] $\equiv . \sim p . \vee . q \vee \sim p \vee r :$

[★5·31·37] $\equiv : \sim p . \vee . \sim p \vee q \vee r :$

[★5·33] $\equiv : \sim p \vee \sim p . \vee . q \vee r :$

[★5·25] $\equiv : \sim p . \vee . q \vee r :$

[★5·55] $\equiv : p . \supset . q \vee r :. \supset \vdash . \text{Prop.}$

·79 $\vdash :. q \supset p . \vee . r \supset p : \equiv : q . r . \supset . p$

Dem.

$\vdash . \star 5{\cdot}1{\cdot}39 . \supset \vdash :. q \supset p . \vee . r \supset p : \equiv : \sim p \supset \sim q . \vee . \sim p \supset \sim r :$

[★5·78] $\equiv : \sim p . \supset . \sim q \vee \sim r :$

[★5·6 . ★4·61] $\equiv : \sim p . \supset . \sim (q . r) :$

[★5·1] $\equiv : q . r . \supset . p :. \supset \vdash . \text{Prop.}$

Note. The analogues, for classes, of ★5·78·79 are false. Take, *e.g.*, ★5·78, and put $p =$ English people, $q =$ men, $r =$ women. Then p is contained in q or r, but is not contained in q and is not contained in r.

★5·8 $\vdash : p \supset \sim p . \equiv . \sim p$ [★2·91 . Simp.]

·81 $\vdash : \sim p \supset p . \equiv . p$ [★3·41 . Simp.]

·82 $\vdash : p \supset q . p \supset \sim q . \equiv . \sim p$ [★3·44 . Imp. ★3·31 . Comp.]

·83 $\vdash : p \supset q . \sim p \supset q . \equiv . q$ [★3·42 . Imp. Simp. Comp.]

Note. ★5·82·83 may also be obtained from ★5·43, of which they are virtually other forms.

★6· *Miscellaneous propositions.*

The following are mainly propositions inserted on account of their subsequent utility. They are all easy to prove, and I shall therefore either wholly omit the proofs, or merely indicate the propositions used in the proofs.

·1 $\vdash : p . \vee . p \supset q$ [★3·31]

·11 $\vdash : p \supset q . \vee . \sim p \supset q$ [★3·35]

·12 $\vdash : p \supset q . \vee . p \supset \sim q$ [★3·34]

·13 $\vdash : p \supset q . \vee . q \supset p$ [★3·32 . Simp.]

·14 $\vdash : p \supset q . \vee . q \supset r$ [★3·33·31]

·15 $\vdash : p \equiv q . \vee . p \equiv \sim q$

·16 $\vdash . \sim (p \equiv q . p \equiv \sim q)$

·17 $\vdash : p \vee q . \sim (p . q) . \equiv . p \equiv \sim q$

·18 $\vdash : p \equiv q . \equiv . \sim (p \equiv \sim q)$ [★6·15·16·17]

·19 $\vdash : \sim (p \equiv \sim p)$ $\left[\star 6{\cdot}18\,\frac{p}{q}\,.\ \star 5{\cdot}2\right]$

★6·2 $\vdash : p . q . \supset . p \equiv q$ [★4·4·22]

·21 $\vdash : \sim p . \sim q . \supset . p \equiv q$ [★6·2 . ★5·11]

·22 $\vdash :. \sim (p \equiv q) . \equiv : p . \sim q . \vee . \sim p . q$ [★5·6·54]

·23 $\vdash :. \ p \equiv q . \equiv : p . q . \vee . \sim p . \sim q$ $\left[\star 6{\cdot}18 .\ \star 6{\cdot}22\,\frac{\sim q}{q}\right]$

·24 $\vdash :. \sim (p . q . \vee . \sim p . \sim q) . \equiv : p . \sim q . \vee . \sim p . q$ [★6·22·23]

·3 $\vdash :. p . q . \supset . r : \equiv : p . q . \supset . p . r$

·31 $\vdash :. r . p \supset q : \supset : p . \supset . q . r$

·32 $\vdash :. p . \supset . q \equiv r : \equiv : p . q . \equiv . p . r$ [★5·76 . ★4·32 . ★6·3]

This proposition is constantly required in subsequent proofs.

·33 $\vdash :. p . q \supset r . \equiv : p : p . q . \supset . r$

·34 $\vdash :. p \supset q . \supset : p . q \supset r . \supset . r$

·35 $\vdash :. p \supset q . p \supset r . \supset : p . \supset . q \equiv r$

·36 $\vdash : p . p \equiv q . \equiv . q . p \equiv q$

★6·4 $\vdash :. p . \supset . p \supset q : \equiv . p \supset q$ [Simp. ★3·5]

·41 $\vdash :. p \supset q . \supset . p \supset r : \equiv : p . \supset . q \supset r$ [★3·62·6]

·42 $\vdash :: p . \supset . q \supset r : \equiv :. p . \supset : q . \supset . p . r$ [★6·3 . Exp.]

·43 $\vdash :: p \supset s :. p . \supset : q . \equiv . r . s :. \supset : p . \supset . q \equiv r$

·44 $\vdash :: p \supset q . \supset :. p \supset r . \equiv : p . \supset . q . r$

·5 $\vdash :. p . \supset : p \supset q . \equiv . q$

·51 $\vdash :. p \supset q . \supset : p . \supset . q \vee r$

·52 $\vdash :. q . \vee . p \supset q : \equiv . p \supset q$

·53 $\vdash :. p \vee q \vee r . \supset . s : \equiv : p \supset s . q \supset s . r \supset s$ [★5·77]

·54 $\vdash :. p . q . \equiv . p : \vee : p . q . \equiv . q$

·55 $\vdash :. p \vee q . \equiv . p : \vee : p \vee q . \equiv . q$

·6 $\vdash :. p . \sim q . \supset . r : \equiv : p . \supset . q \vee r$

·61 $\vdash : p \vee q . \sim q . \equiv . p . \sim q$

·62 $\vdash :. p . q . \vee . \sim q : \equiv . p \vee \sim q$ $\left[★5·7 \dfrac{q, p}{p, q} \right]$

·7 $\vdash :. p \vee r . \equiv . q \vee r : \equiv : r . \vee . p \equiv q$

·71 $\vdash ::. r \supset p . \supset :. s . \sim p . \vee . q : \supset . r :. \equiv . s \supset p . q \supset r$

★7· *Propositions Concerning All Values of the Variables.*

In this section, we have to introduce a new primitive idea, namely, the idea "$(C)(x)$ is true for all values of x." Our formulæ hitherto have concerned *any* value of the variable, not *all* values, and what we may call the *range* of the variable has always been the whole of the sentence (*i. e.*, the whole of one asserted proposition, or of as many asserted propositions as enter into a single chain of deductions). The point of the new primitive idea may be illustrated as follows: Consider ★3·31, *i.e.*, $\vdash : \sim p . \supset . p \supset q$. This means: "For any values of p and q, not-p implies that $p \supset q$". Here although q only occurs at the end, its "range" is the whole proposition. But suppose we wish to say: "Not-p implies that, for any value of q, p implies q". Here q has as its range only "p implies q", not the whole proposition. We may proceed a stage further, and say: "Not-p implies that p implies that, for any value of q, q is true." Here the range of q is confined to q itself. When the range of (say) q is not the whole of an asserted proposition, we indicate the fact by putting (q) before the beginning of the range, followed by as many dots as occur at the end of the range, or (if the end of the range is the end of the sentence) by one more dot than occurs anywhere between the beginning of the range and the end of the sentence. Thus "$\vdash : \sim p . \supset . (q) . p \supset q$" means: "For any value of p, not-p implies that, for all values of q, p implies q". Similarly, "$\vdash :. \sim p . \supset : p . \supset . (q) . q$" means: "For any value of p, not-p implies that p implies that, for all values of q, q is true". When the assertion-sign is absent, any complex $(C \between x)$, containing x, is a function of x, which in general has different values for different values of x. But the proposition "$(C \between x)$ is true for all values of x" is not a function of x; the result, so far as x is concerned, is a constant. [The case is more or less like that of the x in $\int_a^b f(x)\,dx$.] Hence we may, following Peano, call the x only an *apparent* variable in such a case. A variable which is not apparent is called *real*. [The sense of *real* which is opposed to *complex* cannot well occur in a context in which the above sense might be meant.] We denote by

$$(x) . (C \between x) \text{ or } (x) : (C \between x)$$

(or by a larger number of dots, if necessary) the proposition "$(C \between x)$ is true for *all* values of x". Thus the "range" of an apparent variable becomes that proposition (whatever it is) whose truth for all values of the variable is being considered.

The utility of the above idea and notation arises when the proposition $(x) . (C \between x)$ is not itself asserted, but is part of an asserted proposition. Thus in the above case, in

$$\vdash :. \sim p . \supset : p . \supset . (q) . q,$$

$(q) . q$ is not itself asserted, but is a constituent of an asserted proposition. Note that $(q) . q$ is an absolute constant, meaning "everything is true". Thus the formula

$$\vdash :. \sim p . \supset : p . \supset . (q) . q$$

may be read: "If p is false, then p implies that everything is true."

In $(C \between x)$, where the x is undetermined, the x is supposed to have some value, but not this or that definite value. Thus, when we assert $(C \between x)$, we assert an ambiguous proposition; this assertion is true if all the values of $(C \between x)$ are true, but not otherwise. This is what is asserted in ★7·1·11 below. For the sake of definiteness, we may say that

"$\vdash . (C \between x)$" means "$(C \between x)$ is true, where x may be anything".

"$\vdash . (x) . (C \between x)$" means "$(C \between x)$ is true for all values of x".

The necessity for this rather subtle distinction arises from certain difficulties concerning deduction with variables, which need not be here considered, as well as from the necessity of a notation for the case where the "range" of a variable is less than the whole of one asserted proposition.*

We may now proceed to the formal development, which will make the above points clearer.

★7·0. *Primitive idea:* "$(x) . (C \between x)$" means "the truth of $(C \between x)$ for all values of x."

·01 $(x, y) . (C \between x, y) . = : (x) : (y) . (C \between x, y)$ Df.

i.e., "$(x, y) . (C \between x, y)$" means "the truth for all values of x, of the truth, for all values of y, of $(C \between x, y)$"; *i.e.*, practically, "the truth of $(C \between x, y)$ for all values of x and y". A similar form will be used, when required, for more than two variables.

·02 $(A \between x) . \supset_x . (B \between x) : = : (x) : (A \between x) . \supset . (B \between x)$ Df.

Peano employs exclusively the notation $(A \between x) . \supset_x . (B \between x)$, which expresses what I call a "formal implication". He has no means of expressing $(x) . (C \between x)$ except when $(C \between x)$ is of the form $(A \between x) . \supset . (B \between x)$. In this respect Frege is preferable. But the above subscript notation is shorter, and often convenient, where implications are concerned; it may therefore be retained with advantage for use on suitable occasions.

·03 $(A \between x, y) . \supset_{x, y} . (B \between x, y) : = : (x, y) : (A \between x, y) . \supset . (B \between x, y)$ Df.

A similar form may be used for more than two variables.

★7·04 $\sim (x) . (C \between x) . = . \sim \{ (x) . (C \between x) \}$ Df.

This definition serves only for the avoidance of brackets. A similar notation will be used for several variables.

·1 $\vdash : (x) . (C \between x) . \supset . (C \between y)$ Pp.

This proposition may be read: "What is true of all is true of any."

·11 What is true of any is true of all. Pp.

---

* The possibility of taking "all values" of a variable, and the nature of the total set of values concerned, are subject to some rather complicated restrictions, not here considered. This results from the paradox of the Cretan who said that all Cretans are liars, from Burali-Forti's paradox, ["Una questione sui numeri transfiniti", Rendiconti del Circolo Matematico di Palermo, Tomo XI, 1897], from the paradox considered in my "Principles of Mathematics", chap. X, and from other analogous paradoxes. As an example of the limitations, "all values of C" in a statement about $(C \between x)$ would be meaningless.

This proposition is the converse of ★7·1. It cannot be symbolized without inventing a new symbol for the purpose, which seems not worth while, as this symbol would not be afterwards required. It might be supposed at first that what we mean could be expressed by

$$“\vdash : (C \between y) . \supset . (a) . (C \between x)”.$$

But this would mean: “$(C \between y)$ implies that $(C \between x)$ is true for all values of x, where y may be anything”, which is not what we mean, and is not in general true. What we mean is: “If $(C \between y)$ is true whatever y may be, then $(C \between x)$ is true for all values of x”. When $(C \between y)$ is asserted, we have

$$\vdash . (C \between y) . \supset \vdash . (x) . (C \between x)$$

but we cannot adopt this as our formula, because we must only put the assertion-sign before what really is true, and here $(C \between y)$ is to be only a hypothesis, which may or may not be true.

★7·12 $\vdash :. (x) . p \supset (C \between x) . \supset : p . \supset . (x) . (C \between x)$ Pp.

i. e., “If it is true, for all values of x, that p implies $(C \between x)$, then p implies that $(C \between x)$ is true for all values of x”.

We now have all the apparatus necessary for deductions such as occur in the ordinary syllogism, *e. g.*, for proving

“If all Greeks are men, and all men are mortals, then all Greeks are mortals”.

The symbolic statement of “all Greeks are men” is

“$(x) : x$ is a Greek. $\supset . x$ is a man”.

Thus the general form of Barbara is

$$\vdash :. (x) . (A \between x) \supset (B \between x) : (x) . (B \between x) \supset (C \between x) : \supset : (x) : (A \between x) \supset (C \between x) .$$

This can be deduced from the above premisses; but for the present I shall confine myself to cases in which there are no undetermined complexes, such as $(A \between x)$, $(B \between x)$, $(C \between x)$. The additional apparatus now available leads to certain further propositions, of great importance, in the theory of implication, and it is only these that concern us now.

★7·13. If a is any constant, $\{(x) . (C \between x)\}$ implies $(C \between a)$.

This is not a primitive proposition, but a consequence of ★2·2 and ★7·1. On the impossibility of expressing it symbolically, the same remarks apply as were made concerning ★2·2.

★7·14. $\vdash : (x) . (C \between x) . \supset . (C \between (A \between x))$ [★7·1·★2·3]

Although ★2·3 could not be symbolized with our symbols, ★7·14 can be, because we are now able to indicate the limited *range* of the x.

·15 $\vdash :. (x) . p \supset (C \between x) . \equiv : p . \supset . (x) . (C \between x)$

Dem.

$\vdash . ★7·12 . \quad \supset \vdash :. (x) . p \supset (C \between x) . \supset : p . \supset . (x) . (C \between x)$ (1)

$\vdash . ★3·12 . \quad \supset \vdash :: (x) . (C \between x) . \supset . (C \between y) : \supset :. p . \supset . (x) . (C \between x) :$
$\supset : p . \supset . (C \between y)$ (2)

$\vdash . (2) . ★7·1 . \supset \vdash :. p . \supset . (x) . (C \between x) : \supset : p . \supset . (C \between y)$

$[★7·11] \quad \supset \vdash :. (x) :. p . \supset . (x) . (C \between x) : \supset : p . \supset . (C \between x) :.$

$[★7·12] \quad \supset \vdash :. p . \supset . (x) . (C \between x) : \supset : (x) : p . \supset . (C \between x)$ (3)

$\vdash . (1) . (3) . \quad \supset \vdash .$ Prop.

Note that in the last two lines of the above proof two different x's occur. In the last line but one, the x which has come down from the previous line has $(C \between x)$ for the whole of its range; the other x replaces the y of the previous line, in accordance with ★7·11, and has the whole proposition for its range. The practice of using two different x's in one proposition is not in general commendable, for in *practice* it is liable to cause confusion; but the system of dots makes confusions *theoretically* impossible, *i. e.*, the meaning is never ambiguous, tho' it may be a little difficult to discover. I shall, however, in future avoid using the same letter for different apparent variables, except where doing so conduces to clearness. Different *real* variables must never be represented by the same letter.

The effect of the above primitive propositions may be stated in certain rules.

1. Given any *asserted* proposition containing a real variable x, we may put an x in brackets at the beginning, with a sufficient number of dots to cover the whole proposition. *E.g.*, given $\vdash : x \equiv y . y \equiv z . \supset . x \equiv z$, we may infer

$$\vdash : (x) : x \equiv y . y \equiv z . \supset . x \equiv z$$

2. Given any proposition beginning by an x in brackets, followed by a sufficient number of dots to cover the whole proposition, we may infer the proposition which results from omitting the x in brackets.

3. Given an implication which begins with an x in brackets, followed by a sufficient number of dots to cover the whole proposition, then, if the hypothesis does not contain x, we may remove the x in brackets to the beginning of the apodosis.

4. Given an *asserted* implication, containing x in the apodosis, but not in the hypothesis, we may insert at the beginning of the apodosis an x in brackets, followed by a sufficient number of dots to cover the whole of the apodosis.

★7·2 $\vdash :.\ p\ .\equiv :\ (s)\ .\ s \supset p$

This proposition states that p implies and is implied by the proposition "for all values of s, s implies p"; *i. e.*, p implies and is implied by the proposition "everything implies p".

Dem.

$\vdash$. Simp. $\quad \supset \vdash :.\ p\ .\supset .\ s \supset p$ (1)

$\vdash$. (1) . ★7·11 . $\quad \supset \vdash :.\ (s) :\ p\ .\supset .\ s \supset p$ (2)

$\vdash$. ★7·12 . $\quad \supset \vdash ::\ (s) :\ p\ .\supset .\ s \supset p :\supset :.\ p\ .\supset :\ (s)\ .\ s \supset p$ (3)

$\vdash$. (2) . (3) . $\quad \supset \vdash :.\ p\ .\supset :\ (s)\ .\ s \supset p$ (4)

$\vdash$. ★7·14 . $\quad \supset \vdash :.\ (s)\ .\ s \supset p\ .\supset .\ \sim p \supset p\ .$

[★3·41] $\quad \supset .\ p$ (5)

$\vdash$. (4) . (5) . $\quad \supset \vdash$. Prop.

★7·21 $\vdash :\ \sim p\ .\equiv .\ (s)\ .\ p \supset s$

i. e., the falsehood of p implies and is implied by the proposition "for all values of s, p implies s".

Dem.

$\vdash$. ★3·31 . $\quad \supset \vdash :\ \sim p\ .\supset .\ p \supset s$ (1)

$\vdash$. (1) . ★7·11 . $\quad \supset \vdash :\ (s) :\ \sim p\ .\supset .\ p \supset s :$

[★7·12] $\quad \supset \vdash :\ \sim p\ .\supset .\ (s)\ .\ p \supset s$ (2)

$\vdash$. ★7·14 . $\quad \supset \vdash :\ (s)\ .\ p \supset s\ .\supset .\ p \supset \sim p\ .$

[★2·91] $\quad \supset .\ \sim p$ (3)

$\vdash$. (2) . (3) . $\quad \supset \vdash$. Prop.

Note. ★7·11 and ★7·12 are constantly being used together. They enable us, from any *asserted* proposition of the form

$\vdash :\ (C \between x)\ .\supset .\ (A \between x, y)$ [x and y being undetermined]

to infer $\vdash :\ (C \between x)\ .\supset .\ (y)\ .\ (A \between x, y)$.

In order to apply ★7·11, it is necessary, in such a case, to take the *whole* proposition $(C \between x) . \supset . (A \between x, y)$, not only $(A \between x, y)$. For $(A \between x, y)$ is unasserted, and therefore does not necessarily imply $(y) . (A \between x, y)$. For this reason it would be a fallacy to write

$$\begin{aligned} &\vdash : (C \between x) . \quad \supset . (A \between x, y) . \\ &[\bigstar 7 \cdot 11 \cdot 12] \supset . (y) . (A \between x, y) \end{aligned}$$

although our conclusion, namely $(C \between x) . \supset . (y) . (A \between x, y)$, would be true.

★7·22 $\vdash :. \sim p . \equiv : p . \supset . (s) . s$

Dem.

$\vdash . \bigstar 7 \cdot 15 . \quad \supset \vdash :. (s) . p \supset s . \equiv : p . \supset . (s) . s$ (1)

$\vdash . (1) . \bigstar 7 \cdot 21 . \supset \vdash .$ Prop.

Thus p is false when, and only when, p implies the proposition "everything is true".

·23 $\vdash :. \sim p . \equiv : p . \equiv . (s) . s$

Dem.

$\vdash . \bigstar 7 \cdot 14 . \quad \supset \vdash : (s) . s . \supset . p :$

$[\bigstar 5 \cdot 73] \quad \supset \vdash :. p . \supset . (s) . s : \equiv : p . \equiv . (s) . s$ (1)

$\vdash . (1) . \bigstar 7 \cdot 22 . \supset \vdash .$ Prop.

·24 $\vdash :. p . \equiv : p . \equiv . \sim (s) . s$

Dem.

$\vdash . \bigstar 7 \cdot 23 \frac{\sim p}{p} . \supset \vdash :. p . \equiv : \sim p . \equiv . (s) . s :$

$[\bigstar 5 \cdot 12] \quad \equiv : p . \equiv . \sim (s) . s :. \supset \vdash .$ Prop.

Thus both p and not-p may be replaced, in implicational formulæ, by equivalences. The above propositions ★7·23·24 may be read as follows: To say that p is false is equivalent to saying that p is equivalent to "everything is true"; and to say that p is true is equivalent to saying that p is equivalent to "not everything is true".

★7·25 $\vdash . \sim (s) . s$

i.e., "not everything is true".*

Dem.

$\vdash . \text{Id.} \supset \vdash : (s) . s . \supset . (s) . s :$

$\left[★7{\cdot}22 \frac{(s) . s}{p} \right] \supset \vdash : \sim (s) . s : \supset \vdash . \text{Prop.}$

·26 $\vdash . \sim \{ (s) . s . \equiv . \sim (s) . s \}$ $\left[★6{\cdot}19 \frac{(s) . s}{p} \right]$

Note that ★2·2 is used here, not ★2·3.

·27 $\vdash : (s) . s . \supset . p$ [★7·1]

·28 $\vdash : p . \supset . \sim (s) . s$ [★7·25 . Simp.]

Note. $(s) . s$ and $\sim (s) . s$ are absolute constants, which play a part analogous to that of 0 and 1 in ordinary algebra. The analogy is illustrated by ★7·3·31·33 below; but ★7·32 makes $\sim (s) . s$ analogous rather to ∞ than to 1.

★7·3 $\vdash :. p . \equiv : p . \vee . (s) . s$ [★5·74 . ★7·25]

·31 $\vdash :. p . \equiv : p : \sim (s) . s$ [★5·73 . ★7·25]

·32 $\vdash :. \sim (s) . s . \equiv : p . \vee . \sim (s) . s$ [★5·72 . ★7·28]

·33 $\vdash :. (s) . s . \equiv : p : (s) . s$ [★5·71 . ★7·27]

·34 $\vdash : p . \sim p . \equiv . (s) . s$ $\left[★4{\cdot}24 . ★7{\cdot}23 \frac{p . \sim p}{p} \right]$

·35 $\vdash : p \vee \sim p . \equiv . \sim (s) . s$ $\left[★4{\cdot}25 . ★7{\cdot}24 \frac{p \vee \sim p}{p} \right]$

★7·4 $\vdash :. p . \equiv : p \supset s . \supset_s . s$

Dem.

$\vdash . ★3{\cdot}1 . \quad \supset \vdash :. p . \supset : p \supset s . \supset . s :.$

$[★7{\cdot}11] \quad \supset \vdash :. (s) :. p . \supset : p \supset s . \supset . s :.$

$[★7{\cdot}12] \quad \supset \vdash :. p . \supset : p \supset s . \supset_s . s$ (1)

$\vdash . ★7{\cdot}14 . \quad \supset \vdash :. p \supset s . \supset_s . s : \supset : p \supset p . \supset . p :$

$\left[★3{\cdot}51 \frac{p}{q} \right] \quad \supset : p$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop.}$

The above proposition may be stated in words as follows: p is true when and only when everything that p implies is true.

---

* "Everything is not true" would be "$(s) . \sim s$".

★7·41 $\vdash :: p . q . \equiv :. p . \supset . q \supset s : \underset{s}{\supset} . s$

Dem.

$\vdash$. Imp. $\supset \vdash :: p . \supset . q \supset s : \supset : p . q . \supset . s ::$

[Comm] $\supset \vdash :: p . q . \supset :. p . \supset . q \supset s : \supset . s ::$

[★7·11] $\supset \vdash :: (s) :: p . q . \supset :. p . \supset . q \supset s : \supset . s ::$

[★7·12] $\supset \vdash :: p . q . \supset :: p . \supset . q \supset s : \underset{s}{\supset} . s$ (1)

$\vdash$. ★7·14. $\supset \vdash ::. p . \supset . q \supset s : \underset{s}{\supset} . s :. \supset :: p . \supset . q . \supset . p . q :. \supset . p . q ::$

[★4·2] $\supset :: p . q$ (2)

$\vdash$. (1) . (2) . $\supset \vdash$. Prop.

★7·42 $\vdash :: p . q . \equiv :. p \supset s . \vee . q \supset s : \underset{s}{\supset} . s$

Dem.

$\vdash$. ★5·79. $\supset \vdash :: p \supset s . \vee . q \supset s : \supset : p . q . \supset . s ::$

[Comm] $\supset \vdash :: p . q . \supset :. p \supset s . \vee . q \supset s : \supset . s ::$

[★7·11·12] $\supset \vdash :: p . q . \supset :. p \supset s . \vee . q \supset s : \underset{s}{\supset} . s$ (1)

$\vdash$. ★7·14. $\supset \vdash :: p \supset s . \vee . q \supset s : \underset{s}{\supset} . s :. \supset$

$:. p . \supset . p . q : \vee : q . \supset . p . q : \supset . p . q$ (2)

$\vdash$. ★6·54. $\supset \vdash : p . \supset . p . q : \vee : q . \supset . p . q$ (3)

$\vdash$. (2) . (3) . $\supset \vdash :: p \supset s . \vee . q \supset s : \underset{s}{\supset} . s :. \supset . p . q$ (4)

$\vdash$. (1) . (4) . $\supset \vdash$. Prop.

·43 $\vdash :. p . q . r . \equiv :. p \supset s . \vee . q \supset s . \vee . r \supset s : \underset{s}{\supset} . s$ [★7·42]

·5 $\vdash :. p \vee q . \equiv : p \supset s . q \supset s . \underset{s}{\supset} . s$ [★5·77]

The proof proceeds on the same lines as the proof of ★7·42.

·51 $\vdash :. p \vee q \vee r . \equiv : p \supset s . q \supset s . r \supset s . \underset{s}{\supset} . s$ [★7·5]

Note. Instead of taking negation as a primitive idea, as was done in ★1, it is possible to regard the property stated in ★7·22 as giving the *definition* of negation, *i. e.*, we may put

$$\sim p . = : p . \supset . (s) . s \qquad \text{Df.}$$

This requires that the primitive idea $(x) . (C \between x)$ should be introduced in ★2, and not postponed till ★7. Instead of the definitions in ★4·1·11 it is now simpler to take ★7·41 and ★4·5 as the definitions of $p\,.\,q$ and $p \vee q$ respectively, putting

$$p\,.\,q\,. = :.\, p\,.\supset.\,q \supset s :\supset_s .\, s \qquad \text{Df.}$$

$$p \vee q\,. = : p \supset q\,.\supset.\,q \qquad \text{Df.}$$

Instead of ★2·9·91·92, we can now substitute a single primitive proposition, namely

$$\vdash :.\, p \supset q\,.\supset.\,p :\supset.\,p \qquad \text{Pp.*}$$

(which is ★3·51 in the above).

This method is more artificial and much more difficult than the method adopted above, but these disadvantages are more than outweighed by the fact that we have one less primitive idea and two less primitive propositions.

My reason for not adopting this method is not its artificiality or its difficulty, but the fact that it never enables us to know that anything whatever is *false*. It enables us to prove the *truth* of whatever can be proved true by the method adopted above, and it does not enable us to prove the truth of anything which in fact is false. It even enables us to prove, concerning all the propositions which can be proved false by the above method, that, if they are true, then everything is true; but if any man is so credulous as to believe that everything is true, then the method in question is powerless to refute him. For example, we get the law of contradiction in the form

$$\vdash : p\,.\sim p\,.\supset.\,(s)\,.\,s\,;$$

but this does not show that $p\,.\sim p$ is false, unless we assume that $(s)\,.\,s$ is false. Now in the system considered, *falsehood* is not among the ideas that occur in our apparatus; hence we cannot assume that $(s)\,.\,s$ is false without introducing a new primitive idea. But when once we have introduced this new idea, it is economical to make all possible use of it; and this leads us to the method adopted above.

The above propositions give what is most important in the theory of *material* implication, *i. e.*, of propositions of the form

$$p \supset q\,.$$

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* This is the principle called "reduction" in my "Principles of Mathematics", §18, (10), p. 17.

The subject which comes next in logical order is the theory of *formal* implication, *i. e.*, of propositions of the form

$$(x):(A \between x) . \supset . (B \between x) .$$

These have been touched on in ★7, but are not there considered on their own account. It is only after these have been considered that we can advance to the theory of classes, and thence to the theory of relations. The necessity of avoiding the contradiction of Burali–Forti concerning the greatest ordinal, and a class of contradictions of which the simplest is discussed in my "Principles of Mathematics", Chap. X, necessitate certain distinctions which render the subsequent development somewhat less simple than it would otherwise be. By distinguishing different *types* of variables, and confining the notion of "all values of the variable" to all values within the type concerned, these contradictions can, I believe, be all satisfactorily avoided.

The Geometry of Differential Elements of the Second Order with Respect to the Group of all Point Transformations.

BY EDWARD KASNER.

A simple and familiar example of an expression which is unaltered by arbitrary point transformation of the plane is given by the anharmonic ratio of the tangents to four curves passing through a common point. It is understood that the functions defining the transformation

$$X = \phi(x, y), \quad Y = \psi(x, y) \tag{1}$$

possess first derivatives and that the Jacobian

$$J \equiv \phi_x \psi_y - \phi_y \psi_x \tag{2}$$

does not vanish at the point in question. The invariant property here results from the fact that, whatever the form of ϕ and ψ, the lineal elements at a point undergo linear transformation, namely

$$Y' = \frac{\psi_x + y' \psi_y}{\phi_x + y' \phi_y}.$$

In this paper we obtain absolute invariants involving the curvatures as well as the directions of curves passing through a point, *i. e.*, invariants of differential elements of the second order. The transformations of y', y'' are no longer linear, but cubic, and give rise to the eight-parameter group discussed in §1. The invariants are obtained from the infinitesimal generators of this group in §5. They may also be obtained synthetically by means of a fundamental relation, established in §2, between the geometry of differential elements considered with respect to arbitrary point transformations and projective geometry in space of four dimensions. This relation suggests a rational classification of differential equations of the second order (§3). It also furnishes a geometric interpretation for the new invariants obtained (§6). In the last section (§7) the invariants for conformal transformation, which are not all expressible as angles, are obtained by a very simple relation to similitude transformations.

§1.—*The Group G_8.*

Consider a bundle of elements, *i. e.*, the ∞^2 differential elements of the second order at a given point. An element is determined by the values of y', y''. By a point transformation the bundle is turned into a bundle and the coordinates of the new elements are

$$Y' = \frac{\psi_x + y'\psi_y}{\phi_x + y'\phi_y}, \quad Y'' = \frac{\alpha + \beta y' + \gamma y'^2 + \delta y'^3 + Jy''}{(\phi_x + y'\phi_y)^3}, \tag{3}$$

where $\alpha, \beta, \gamma, \delta$ involve the second derivatives of ϕ and ψ.* The coefficients are numbers since the point x, y is fixed and only y', y'' vary. It will be convenient to use the simplified notation

$$u = y', \; v = y''; \; U = Y', \; V = Y''.$$

Our transformation is then of the form

$$U = \frac{au + b}{cu + d}, \quad V = \frac{\alpha + \beta u + \gamma u^2 + \delta u^3 + (ad - bc)v}{(cu + d)^3}, \tag{4}$$

This cannot degenerate since it is assumed that

$$J = ad - bc \neq 0.$$

The constants a, b, c, d, α, β, γ, δ are arbitrary and essential.

The infinite group of arbitrary point transformations thus induces on a bundle of elements the eight-parameter continuous group (4), *which we shall refer to as G_8.*

The eight infinitesimal generators of the group are found to be

$$\frac{\partial}{\partial u}, \; \frac{\partial}{\partial v}, \; u\frac{\partial}{\partial u}, \; v\frac{\partial}{\partial v}, \; u^2\frac{\partial}{\partial u} + 3uv\frac{\partial}{\partial v}; \; u\frac{\partial}{\partial v}, \; u^2\frac{\partial}{\partial v}, \; u^3\frac{\partial}{\partial v}. \tag{5}$$

§2.—*Representation in Four-dimensional Space.*

It will be noted that the numerator and denominator of V in (4) are linear with respect to u, u^2, u^3, v. This suggests the representation of each element

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* It is assumed that these second derivatives exist and that J does not vanish at the point considered.

(u, v) of our bundle by means of a point in four space R_4 with the homogeneous coordinates*

$$z_1 : z_2 : z_3 : z_4 : z_5 = 1 : u : u^2 : u^3 : v \tag{6}$$

The ∞^2 points in R_4 representing the ∞^2 elements then constitute the manifold defined by

$$z_1 z_3 - z_2^2 = 0, \ z_2 z_3 - z_1 z_4 = 0, \ z_3^2 - z_2 z_4 = 0. \tag{7}$$

This is a cubic cone S_3 with its vertex at the point $(0, 0, 0, 0, 1)$; its section in the space $z_5 = 0$ is a twisted cubic.

The transformation (4) now may be written in the following linear form:

$$\left.\begin{aligned} Z_1 &= d^3 z_1 + 3\,cd^2 z_2 + 3c^2 d z_3 + c^3 z_4 \\ Z_2 &= b^2 d z_1 + d(ad + 2bc) z_2 + c(bc + 2ad) z_3 + ac^2 z_4 \\ Z_3 &= bd^2 z_1 + b(bc + 2ad) z_2 + a(ad + 2bc) z_3 + a^2 c z_4 \\ Z_4 &= b^3 z_1 + 3ab^2 z_2 + 3a^2 b z_3 + a^3 z_4 \\ Z_5 &= \alpha z_1 + \beta z_2 + \gamma z_3 + \delta z_4 + (ad - bc) z_5 \end{aligned}\right\} \tag{8}$$

This leaves the relations (4) invariant.

Thus to each transformation (4) *of the bundle of elements corresponds in R_4 a collineation for which the cubic surface S_3 is invariant.*

It is easy to show in fact that a cubic cone in R_4 admits no larger continuous group of collineation. The general collineation of R_4 involves 24 constants. Since the vertex of the cone must be a fixed point, these are reduced to 20. Furthermore, the twisted cubic in the plane $z_5 = 0$ must be converted into one of the ∞^4 twisted cubics on the cone. Since the total number of twisted cubics in R_4 is ∞^{16}, this imposes $16 - 4 = 12$ restrictions. Hence the automorphic group in question contains precisely $20 - 12 = 8$ parameters.

We have thus obtained the following principle:

The geometry of a bundle of differential elements of the second order in a plane, with respect to arbitrary point transformation, is equivalent to projective geometry on a cubic cone in a space of four dimensions.

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*The z's may of course be considered directly as a system of redundant coordinates for defining an element of a bundle. This is the natural system for the group of all point transformations. For the projective group of the plane, an appropriate system has been introduced by E. Study (Leipziger Berichte, 1901).

§3.—*Classification of Differential Equations of the Second Order.*

This principle suggests the classification of differential equations of the second order which are algebraic in y', y'', that is in u, v, according to the degree of the corresponding equation expressed in the z coordinates. To an equation of first degree

$$A_1 z_1 + A_2 z_2 + A_3 z_3 + A_4 z_4 + A_5 z_5 = 0,$$

corresponds in terms of u, v an equation

$$A_1 + A_2 u + A_3 u^2 + A_4 u^3 + A_5 v = 0.$$

If now the coefficients are taken as arbitrary functions of x, y, we have the differential equations of the form

$$Ey'' = Ay'^3 + By'^2 + Cy' + D. \tag{9}$$

This is the type which has been extensively studied by Lie, R. Liouville, and Tresse. It includes the differential equation of the geodesics on any surface.

The next case, corresponding to an equation of second degree in the z's, leads to the equations

$$\begin{aligned} Ay''^2 &+ (B_1 + B_1 y' + B_2 y'^2 + B_3 y'^3)y'' \\ &+ (C + C_1 y' + C_2 y'^2 + C_3 y'^3 + C_4 y'^4 + C_5 y'^5 + C_6 y'^6) = 0 \end{aligned} \tag{10}$$

where the coefficients are arbitrary functions of x, y.

In general we define the *rank* of a differential equation which is algebraic with respect to y' and y'' as the degree of the corresponding z equation.

The rank of a differential equation is invariant under arbitrary point transformation.

The equations of second rank, given by (10), involve 11 arbitrary coefficients instead of the 14 contained in a general quadric in R_4. This is of course due to the three quadratic identities (7). The most general equation of rank r involves

$$\frac{(r+1)(r+2)(r+3)(r+4)}{1\,.\,2\,.\,3\,.\,4} - 3\frac{(r-1)\,r\,(r+1)(r+2)}{1\,.\,2\,.\,3\,.\,4} - 1$$

arbitrary functions of x, y.

Equality of rank is necessary, but not sufficient for the equivalence of two differential equations under point transformation. Thus, among the equations of second rank there is an invariant species defined by $A = 0$; this arises when the quadric manifold in R_4 passes through the vertex of the cone. Subspecies may then be distinguished according to the number of generators of the cone which lie in the quadric. Thus for two generators we find the type

$$(B + B_1 y')\, y'' + (C + C_1 y' + C_2 y'^2 + C_3 y'^3 + C_4 y'^4) = 0, \tag{11}$$

which is thus invariant under point transformation.

In order that two differential equations of the second order shall be equivalent under point transformation, it is NECESSARY *that the corresponding curves on the fundamental cone in R_4 shall be projectively equivalent.**

§4.—*The Transformation of Centres.*

To interpret the equations of various ranks geometrically, without leaving the plane, consider for each element (u, v) the corresponding centre of curvature

$$x_0 = -\frac{u(1+u^2)}{v}, \quad y_0 = \frac{1+u^2}{v}. \tag{12}$$

A point transformation, operating on the elements according to (4), induces the following transformation of the centres:

$$\left.\begin{aligned} X_0 &= \frac{(ax_0 - by_0)\,\{(ax_0 - by_0)^2 + (cx_0 - dy_0)^2\}}{\alpha y_0^3 - \beta x_0 y_0^2 + \gamma x_0^2 y_0 - \delta x_0^3 + (ac - bd)\,(x_0^2 + y_0^2)} \\ Y_0 &= \frac{-(cx_0 - dy_0)\,\{(ax_0 - by_0)^2 + (cx_0 - dy_0)^2\}}{\alpha y_0^3 - \beta x_0 y_0^2 + \gamma x_0^2 y_0 - \delta x_0^3 + (ac - bd)\,(x_0^2 + y_0^2)} \end{aligned}\right\} \tag{13}$$

The centres thus undergo a group of cubic Cremona transformations.

An equation of first rank (9) establishes a relation between x_0 and y_0 of the form

$$E(x_0^2 + y_0^2) = -Ax_0^3 + Bx_0^2 y_0 - Cx_0 y_0^2 + Dy_0^3. \tag{14}$$

In case of equations of first rank, the locus of the centres of curvature of the ∞^1 *integral curves passing through a fixed point is a cubic curve of type* (14).†

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*The invariants of differential equations of the second order have been determined by Tresse in the Preisschriften Jablon. Gesell. zu Leipzig, vol. 32, 1896. Even the equations of first rank have invariants, and are therefore not all equivalent.

†Such a cubic curve is characterized by the fact that the fixed point (the origin) is a double point and the two tangents are the minimal lines.

For the general equation of rank r, the c
order $3r$. If the central locus in the general c
differential equation is a special equation of ra

As remarked above, the differential equa
expressed in arbitrary Gaussian coordinates is

If a surface is represented point by point
point are pictured by curves whose centres of cu
cubic curve of type (14).*

In particular, a pencil of straight lines is c
into a system of curves for which the central

§5.—*Absolute Inva*

Consider any number of elements

$$(u_1, v_1), \quad (u_2, v_2), \quad \ldots$$

A function of these $2n$ arbitrary quantities is
transformations provided it admits the group (
the eight infinitesimal transformations (5). W
of linear partial differential equations:

$$\sum \frac{\partial f}{\partial u_k} = 0, \quad \sum \frac{\partial f}{\partial v_k} = 0, \quad \sum u_k \frac{\partial}{\partial}$$

$$\sum \Big(u_k^2 \frac{\partial f}{\partial u_k} + 3 u_k v_k$$

$$\sum u_k \frac{\partial f}{\partial v_k} = 0, \quad \sum u_k^2 \frac{\partial f}{\partial v_k} =$$

where the summations are extended from $k =$

The first two of the equations (15) show
of the u's and the differences of the v's; wh
homogeneous in these differences. Hence f is
quantities

$$\xi_i = \frac{u_i - u_1}{u_2 - u_1}, \quad \eta_i = \frac{v_i - v_1}{v_2 - v_1}$$

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* If the representation is conformal, the central locus is a
extensive class of representations. See Bull. Amer. Math. Soc.,

ıntroduce these as the independent variables, the last four equations
tem (15) become

$$\left.\begin{array}{l} {}_i(\xi_i - 1) f_{\xi_i} + 3\Sigma\eta_i(\xi_i - 1) f_{\eta_i} = 0, \\ \eta_i - \xi_i) f_{\eta_i} - 0, \quad \Sigma\xi_i(\xi_i - 1) f_{\eta_i} = 0, \quad \Sigma\xi_i^2(\xi_i - 1) f_{\eta_i} = 0, \\ \hfill (i = 3, \ldots, n) \end{array}\right\} \quad (1$$

ıting the first of these equations, we find that f is a function of the $2n$ —
ies

$$\left.\begin{array}{ll} r_i = \dfrac{\xi_3(\xi_i - 1)}{\xi_i(\xi_3 - 1)} & (i = 4, \ldots, n) \\ s_i = \dfrac{\eta_i}{\xi_i^3} & (i = 3, \ldots, n) \end{array}\right\} \quad (17$$

naining equations (16) then become

$$\left.\begin{array}{l} \sum\limits_4^n (s_i - 1) f_{s_i} = 0, \\ f_{s_3} + \sum\limits_4^n r_i f_{s_i} = 0, \\ f_{s_3} + \sum\limits_4^n r_i^2 f_{s_i} = 0. \end{array}\right\} \quad (1$$

ıe first of these equations shows that we may take as the independe
es the $2n - 6$ quantities

$$\left.\begin{array}{ll} \sigma_i = \dfrac{s_i - 1}{s_3 - 1} & (i = 4, \ldots, n) \\ r_i & \end{array}\right\} \quad (18$$

ıer equations then reduce to

$$\left.\begin{array}{l} \sum\limits_4^n (\sigma_i - r_i) f_{s_i} = 0, \\ \sum\limits_4^n (\sigma_i - r_i^2) f_{s_i} = 0. \end{array}\right\} \quad (1$$

ıe next step in the solution brings in the independent variables

$$\left.\begin{array}{ll} \tau_i = \dfrac{\sigma_i - r_i}{\sigma_4 - r_4} & (i = 4, \ldots, n), \\ r_i & \end{array}\right\} \quad (1$$

and reduces the system to the single equation

$$\sum_5^n \{r_i - r_i^2 + \tau_i (r_4^2 - r_4)\} f_{\tau_i} = 0. \tag{19}$$

Integrating this, we find that f is a function of

$$\left.\begin{array}{ll} \Omega_i = \dfrac{(r_4^2 - r_4)\,\tau_i - (r_i^2 - r_i)}{(r_4^2 - r_4)\,\tau_5 - (r_5^2 - r_5)} & (i = 6, \ldots, n) \\ r_i & (i = 4, \ldots, n) \end{array}\right\} \tag{20}$$

Our result then is that

All (absolute) invariants, with respect to the infinite group of point transformations, of n arbitrary differential elements of second order $(u_1, v_1), \ldots, (u_n, v_n)$ *may be expressed as functions of* $2n - 8$ *invariants, namely:* $n - 3$ *anharmonic ratios*

$$r_4, r_5, \ldots, r_n,$$

depending upon only the directions of the elements, and $n - 5$ *of new type*

$$\Omega_6, \Omega_7, \ldots, \Omega_n,$$

which involve the curvatures.

The new type presents itself then only when there are six (or more) elements.*

The system of invariants given is complete not merely in the general functional sense, but also in a special algebraic sense. A detailed discussion, by means of a canonical representation, yields in fact the conclusion:

All RATIONAL *invariants, i. e., all invariants which involve* $u_1, v_1, \ldots, u_n, v_n$ *rationally, are expressible as* RATIONAL *functions of the fundamental invariants of types* r *and* Ω.

§6.—*Geometric Interpretation.*

In order to obtain a geometric interpretation for our absolute invariants, we make use of the corresponding projective geometry in R_4.

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* In his "Théorie des Invariants Universelles" (Jour. de l'École Polytechnique, ser. 2, vol. 4, 1898), P. Rabut attempts to determine all the invariants (relative as well as absolute) of differential elements in general. His method, however, seems to be special, as he does not obtain the above type Ω. In fact, one of his conclusions is that elements whose tangents are all distinct have no absolute invariants beyond those of the familiar type r.

If the group considered is the general projective group of R_4, and points are taken in arbitrary position, then no absolute invariant occurs for fewer than 7 points. In the case of 7 points, we may pass 3-flats through each of the quadruples 1234, 1235, 1236, 1237; these are members of a pencil and thus have an anharmonic ratio which is, of course, an absolute invariant. Our problem, however, is to find the invariants of points situated on the cone S_3 with respect to the special collineation group (8). Here invariants arise for four points. For these determine four generators of the cone, which have an anharmonic ratio, namely, the anharmonic ratio of the four points in which these generators intersect any one of the twisted cubics on S_3. This invariant is of course equivalent to r_4.

Consider now the case of six points on the surface. If we adjoin to these the vertex, (0, 0, 0, 0, 1), of the cone, which is of course invariant for G_8, we have seven points, and so the general projective geometry applies. If the vertex is one of the three points through which 3-flats are passed, we obtain simply one of the r invariants. But if this is not the case, the invariant is of new type.

We can now transfer these results to the plane. Any four elements of a bundle determine an equation of first rank; hence the four centres of curvature determine a cubic curve of type (14).* If now we have six elements at the common point C_0 with centres denoted by C_1, C_2, C_3, C_4, C_5, C_6, we may construct the proper cubics determined by the quadruples

$$C_1\,C_2\,C_3\,C_4,\quad C_1\,C_2\,C_3\,C_5,\quad C_1\,C_2\,C_3\,C_6$$

and the degenerate cubic determined by

$$C_1\,C_2\,C_3\,O,$$

the last consisting of the straight lines connecting C_1, C_2, C_3 with the fixed point O. These cubics are members of a pencil, and hence have an anharmonic ratio I_6, which may be represented concretely by the anharmonic ratio of the tangents at say C_1 to the proper cubics and the line $C_1\,O$.

This absolute invariant I_6 is equivalent to Ω_6; either one may be expressed as a rational function of the other and of invariants of type r. The explicit expressions may be obtained by means of a canonical form, but are not of any particular interest. The complete system obtained in §5 may be replaced by that composed of the types r and I.

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* The cubic degenerates only when two of the elements have a common tangent, or when one of the elements is cuspidal. These cases are explicitly excluded.

§7.—*Conformal Invariants.*

If the point transformation (1) is conformal, the coefficients appearing in the element transformation (4) are connected by the relations

$$a - d = 0, \quad b + c = 0,$$
$$\alpha - \gamma = 0, \quad \beta - \delta = 0.$$

Hence the conformal transformations give rise to a four-parameter sub-group G_4:

$$U = \frac{au + b}{a - bu}, \quad V = \frac{(\alpha + \beta u)(1 + u^2) + (a^2 + b^2)v}{(a - bu)^3}. \tag{21}$$

The infinitesimal generators are

$$v\frac{\partial}{\partial v}, \quad (1 + u^2)\frac{\partial}{\partial v}, \quad u(1 + u^2)\frac{\partial}{\partial v}, \quad (1 + u^2)\frac{\partial}{\partial u} + 3uv\frac{\partial}{\partial v}. \tag{22}$$

The invariants of any number of elements with respect to conformal transformations might be calculated directly from these infinitesimal transformations; but a simple principle, now to be established, will yield them synthetically.

We observe first that if (21) is expressed in terms of the corresponding centres of curvature, we obtain instead of the cubic transformations (13), the linear transformations

$$X_0 = \frac{(a^2 + b^2)(ax_0 - by_0)}{-\beta x_0 + \alpha y_0 + a^2 + b^2}, \quad Y_0 = \frac{(a^2 + b^2)(bx_0 + ay_0)}{-\beta x_0 + \alpha y_0 + a^2 + b^2}. \tag{23}$$

This projective group is composed of those collineations of the plane which leave invariant the origin and the minimal lines passing through it.

If we take the dual of these transformations with respect to the unit circle about the origin, the new transformations leave invariant the line at infinity and the two circular points upon it, and hence are similitude transformations. Hence the conformal group (21) may be related to the similitude group.

We thus obtain a correspondence between the ∞^2 *elements of our bundle and the* ∞^2 *straight lines of the plane. The line corresponding to any given element is the polar of its centre of curvature with respect to the unit circle.* If the differential coordinates of the element are (u, v), the corresponding straight line is

$$u(1 + u^2)x - (1 + u^2)y + v = 0.$$

When the elements are operated on by any conformal transformation, the induced transformation of the straight lines is simply a similitude transformation.

Hence the conformal invariants of any number of elements may be obtained from the similitude invariants of the same number of straight lines.

For two straight lines, the (absolute) invariant is their angle. This of course is also the angle between the elements. For three lines no new type occurs. For four lines there are only three independent angles, while the number of independent invariants is four; the new type is say the ratio of the segments formed by the intersections of three of the lines on the fourth line. For n lines we have $n-1$ angles and $n-3$ independent ratios.*

With respect to the infinite group† *of conformal transformations of the plane, n arbitrary curvature elements of a bundle have $2n-4$ independent absolute invariants; of these $n-1$ are the angles between successive elements and $n-3$ are of new type.*

The latter may be expressed either as ratios, by means of the corresponding straight lines, or as cross-ratios by means of the centres of curvature. If four elements at the common point O have the centres C_1, C_2, C_3, C_4, the new invariant is equivalent to the anharmonic ratio of the lines connecting say C_1 with C_2, C_3, C_4, and O.

The classification of differential equations given in §3 of course remains valid for the conformal sub-group. Equations of the same rank may furthermore be divided into species according to the order of the central loci. When the loci are straight lines, the equations are of the type

$$y'' = (My' + N)(1 + y'^2), \tag{24}$$

which is conformally distinct from the other equations of first rank, though not distinct in the general point group. It may be shown, indeed, that the conformal transformations are the only ones leaving the type (24) invariant.

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* It has been assumed that the conformal equations are of the direct type (not reversing angles) so that the groups considered are continuous. In the reverse type we obtain, instead of (23), a projective transformation which interchanges the minimal lines at 0; the corresponding similitude is then also of the reverse type.

† An examination of the six-parameter group of circular transformations yields the same results. Cf. the author's paper "A relation between the circular and projective transformations of the plane" (Annals of Mathematics, Vol. 5, 1904, pp. 99–104). The invariant described in the closing sentence on page 104 is independent of the angles between the circles.

Gyroscopes and Cyclones.

By F. J. B. Cordeiro.

The most familiar form of gyroscope is the top. The motion of such a body, spinning about an axis of symmetry and actuated by no other force except that of gravity, is easily derived from equations expressing the conservation of the living forces and the conservation of the momental areas about a vertical axis passing through the point of support.

Two equations can thus be obtained by which the angle ϑ—the inclination of the axis to the vertical—and the angle ψ—the azimuth of the plane passing through the axis and the fixed vertical line—can be found by quadratures. These equations were first obtained by Lagrange in his Mecanique Analytique, and were afterwards given by Poisson in his Traite de Mecanique. The former passes them over with but slight notice and proceeds to discuss the small oscillations of a body of any form suspended under the action of gravity from a fixed point. The latter limits the equations to the case in which the body has an initial angular velocity only about its axis, and applies them directly to determine the small oscillations of a top (1) when its axis is nearly vertical, and (2) when its axis makes a nearly constant angle with the vertical. After these necessarily limited results no effort seems to have been made for many years to get a clear conception of the nature of the forces at work. In the preface of Barnard's "Rotary Motion" we find "For some time the impression prevailed in the popular mind that the phenomena exhibited by the apparatus (gyroscope) could not be explained by natural laws. This idea was perhaps strengthened by the name applied to it by Prof. Olmstead (of Yale College), who called it "The Mechanical Paradox."

Quoting further from the same author, "After reading most of the popular explanations of the above phenomena given in our scientific and other publications, I have found none altogether satisfactory. While with more or less success, they expose the more obvious features of the phenomenon and find in the force of gravity an efficient cause of horizontal motion, they usually end in destroying the foundation on which their theory is built, and leave an effect to exist without a cause: a horizontal motion of the revolving disc about a point of support is supposed to be accounted for, while the descending motion, which is the first and direct effect of gravity (without which no horizontal motion can take place) is ignored or supposed to be entirely eliminated. Indeed, it is gravely stated as a distinguishing peculiarity of rotary motion that, while gravity acting upon a non-rotating body causes it to descend vertically, the same force acting upon a rotary body causes it to move horizontally. A tendency to descend is supposed to produce the effect of an actual descent; as if, in mechanics, a mere tendency to motion ever produced any effect whatever without that motion actually taking place."

It was Major (afterwards General) Barnard of the army engineers, who first translated the cryptic, oracular meaning of Poisson's equations into clear everyday English, so that the motions of a gyroscope became clear to the everyday intellect. This was in 1858.

In the rotation of a gyroscope about a fixed axis, centrifugal forces must be developed. This Barnard did not consider in his discussion of the problem. As in the case of a gyroscope with a very rapid rotational motion, the precessional motion is slow, this force will be very small and it can be neglected. But in the theory of the gyroscope in general it must be considered.

Let us consider a gyroscope of the form shown in Fig. 1.

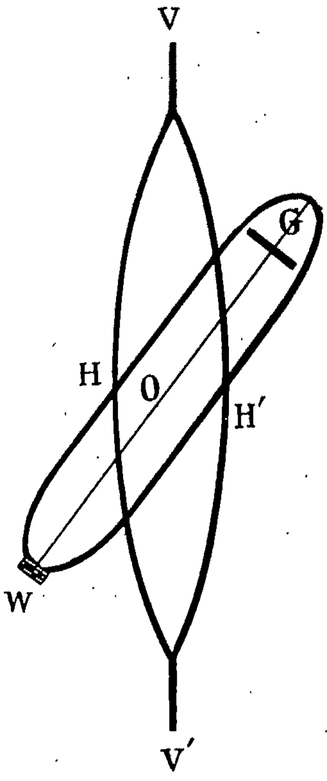

FIG. 1.

The gyroscope G is counterbalanced by the weight W, so that the centre of gravity of the instrument always remains fixed at O. It can turn about a vertical axis VV', and a horizontal axis HH'. Under these circumstances whatever motions are imparted to it, it will not be influenced by gravity.

If an impulsive spin be given to the gyroscope setting up an angular velocity ω, and if it be given an impulsive turn about the axis VV', setting up an angular velocity Ω, its subsequent motion will take place under the action of no external applied forces. Intrinsic forces will be set up, however, modifying its motion continually. If the wheel were not rotating and it were to turn about the axis VV', the centrifugal forces would shortly bring the axis GW to a horizontal position. But if the spin and the turn are both counter-clockwise, a deflective gyroscopic force is set up tending to bring the axis GW into coincidence with the axis VV'. If at the outset of the motion, the component of the centrifugal forces normal to the axis GW, which tends to depress the axis, and the gyroscopic force normal to the axis GW, which tends to raise it, are equal; since they are opposite, there will be no motion of the instrument about the axis HH'.

Let us imagine the equimomental spheroid (Poinsot's momental ellipsoid) at O, with its major axis coincident with the axis GW, rigidly attached to the instrument and rolling on a tangent plane perpendicular to the axis VV'. Such a body will revolve about a line OI connecting the fixed centre O with the point of contact of the spheroid with the tangent plane, as an instantaneous axis. Let the axis GW make an angle γ with the vertical fixed axis OV and an angle i with the moving instantaneous axis OI. Now if the instrument is moving with a uniform angular velocity Ω about the axis OV, and with a uniform angular velocity ω about the axis OG, it will be in the condition of a body moving under the action of no forces, to which an original impulsive couple about the axis OV has been imparted, and one point of which O is fixed. It is shown in treatises on dynamics that under these circumstances certain conditions must hold between the angles in question, γ, i, and the angular velocities about OV and OG. The motion of such a body is exactly represented by causing its momental spheroid to roll on the tangent plane, and the following condition must hold: $\tan\gamma = \frac{A}{C}\tan i$.

Consequently when our gyroscope revolves uniformly about the axis OV, its motion can be represented by causing its momental spheroid to roll upon a perpendicular tangent plane, while its centre is fixed at O.

But we have found another condition requisite for such motion. The centrifugal and gyroscopic forces must balance each other in their turning effort about the axis HH^1. Let us see if these conditions are identical. Drop a perpendicular GM from G to OV. The centrifugal force (exerted in the direction

MG) is $\Sigma mr\Omega^2$. Now the rotation of the gyroscope can clearly have no effect upon this force. Therefore the centrifugal force $= M \,.\, OG \,.\, \sin\gamma\Omega^2$.

The component of this force perpendicular to the axis OG is

$$M \,.\, OG \,.\, \sin\gamma \,.\, \cos\gamma\Omega^2.$$

The gyroscopic force (exerted at right angles to the axis OG) is

$$M \,.\, \frac{k^2\omega \,.\, \Omega \,.\, \sin\gamma}{OG}.$$

These forces are opposed and must be equal. Therefore $OG^2 \,.\, \cos\gamma\Omega = k^2\omega$. But $\Omega\sin\gamma = \omega\tan i$ from the conditions of the motion, where in the above equations M is the mass and A and C are the principal moments about the axis OG. Therefore $\tan\gamma = \frac{A}{C}\tan i$; and we see that the reason that a body under the action of no forces moves about its invariable line at a constant angle is because the centrifugal and gyroscopic forces are in equilibrium about its axis.

Now if at the outset of motion, the angular velocities Ω and ω do not fulfil the condition $\Omega\cos\gamma = \frac{C}{A}\omega$, these forces will not be in equilibrium and the gyroscope will tend to raise or lower itself until such condition is satisfied. The excess of one force over the other will give it a decreasing angular acceleration until it reaches the position of equilibrium. But it will be carried beyond this by its inertia and an increasing acceleration in the opposite direction will bring it to rest and finally back through the position of equilibrium. It will thus oscillate continually through the position of equilibrium as a centre.

The above discussion is of importance in connection with the theory of cyclones. With the establishment of a cyclone and its invariable rotation in a counter-clockwise direction in the Northern Hemisphere, we have nothing to do here. That can be found elsewhere. (See "The Problem of the Cyclone," Monthly Weather Review, November, 1903.) We shall regard a cyclone dynamically as a rigid rotating body—in other words as a gyroscope. That this is permissible is evident from the fact that a stream of water (or air) when directed through a tube curled into the form of a spiral, so that there is a rotating mass, exhibits all the properties of a gyroscope, which is nothing more than rotating matter. We can therefore substitute for any given cyclone a dynamically equivalent gyroscope. For the properties of a gyroscope are derived eventually from the inertia of matter, and whether solid, liquid or gaseous, all matter possesses this essential property of inertia. Now we have used above a coiled spiral tube simply to direct the stream lines, while in a cyclone no such artificial constraint is necessary, since the forces at work in the body of a cyclone preserve the stream lines in a constant relation to each other. We can resolve the motion of the inflowing and outflowing air into two components—one of motion to and from the centre—and the other of circular motion about the centre as an axis. For every component directly towards the centre there is a component away from the centre, so that they eliminate each other. We therefore simply substitute for the circular motion a dynamically equivalent gyroscope—all that is necessary being that the moment of inertia of the two should be equal. And in fact we shall find that the calculated motion of such a gyroscope corresponds with the actual motion found in cyclones.

Let us suppose that a cyclone exists at some point of the Earth's surface and that there are no frictional forces. In other words it is free to preserve a constant rotational energy and to move over the surface of the Earth without resistance. Let us suppose that the rotational and azimuthal angular velocities ω and Ω, are so adjusted that $\tan\gamma = \frac{A}{C}\tan i$, where γ is the angle the axis of the cyclone makes with the axis of the Earth.

In such a case there will be uniform azimuthal motion about the axis of the Earth and the cyclone will preserve a uniform latitude. If the velocity of the Earth at the point underlying the centre of the cyclone is equal to that of the cyclone, the cyclone will remain stationary with respect to the Earth.

If less or greater, then there will be a corresponding motion of the cyclone relatively to the surface of the Earth, but it will move along a parallel of latitude. If ω and Ω do not fulfil the above condition, since the cyclone originally started revolving about the axis of the Earth—*i. e.* the elements of which it is composed had a definite moment about this axis,—it will have to conserve the original moment of momentum about this axis. In order to obtain stable motion therefore, it will strive to put itself in such a position that the invariable angular velocity ω and the variable velocity Ω will assume such a relation that the condition $\tan\gamma = \frac{A}{C}\tan i$ holds, i being the inclination of the instantaneous axis to the axis of the cyclone which varies in the adjustment referred to. Since the relation $\frac{A}{C}\cos\gamma = \frac{\omega}{\Omega}$ holds, if ω is too large relatively to Ω, the axis of the cyclone will move to the North until it adjusts itself; if too small, the axis will move to the South. Now in the great majority of all cyclones ω is too large relatively to Ω, and the general movement is to the Northward. Still there are a very few cyclones which move along a parallel of latitude—at least during the first part of their motion—(Piddington has charted such a cyclone—) and very

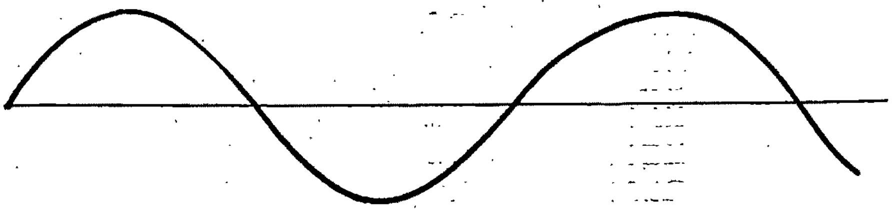

FIG. 2.

fewer still move to the Southward. It is evident that these latter can have very little energy and such we find to be actually the case. A very violent cyclone is apt to move rapidly to the North.

The position of equilibrium of most cyclones is apt to be in high latitudes. As we have seen before, in our experimental gyroscope, this position is apt to be overshot and a succession of oscillations backwards and forwards over this latitude generally occurs. If such an oscillation were developed upon a plane moving surface, the shape of the path would be a series of waves as in Fig. 2.

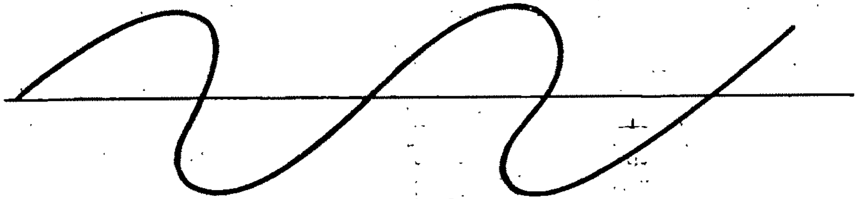

FIG. 3.

But the actual path of a cyclone is developed upon the moving surface of the Earth, and the velocity of the cyclone in higher latitudes is relatively much greater to the velocity of the Earth's surface than it is in lower latitudes. The path therefore will assume the form in Fig. 3.

This is a prediction from purely theoretical considerations. Now on examining the charted paths of cyclones crossing the Atlantic in high latitudes, we find many that exactly execute such a path. If a cyclone could preserve a constant rotational energy, after finding its latitude of equilibrium, it would continue forever circling around the Earth and oscillating about this latitude, in such a path as is pictured in Fig. 3. Abercrombie states that he has traced a cyclone which, starting in the Philippines, went up North to Japan, crossed successively the Pacific, the American continent and the Atlantic Ocean, and then passing over Russia, died out finally on the Steppes of Siberia, having thus almost completely circled the globe.

The writer has previously in a popular article (Bulletin of the American Geographical Society, No. 3, 1900) called attention, he believes for the first time, to the fact that the motion of cyclones is due chiefly to their own intrinsic forces, among which the gyroscopic forces developed are most important. In a later article ("The Problem of the Cyclone," Monthly Weather Review, November, 1903), he sketched out roughly a method of attacking the problem, taking into consideration the effect of frictional forces on the Earth's surface.

The previous explanations of the motions of cyclones, if they can be considered explanations, have been grossly inadequate. We shall mention here only two of these and afterwards consider carefully to what extent, howsoever slightly, they can modify a path which we have shown is due to other and greatly preponderating causes, viz.—the causes we have just discussed together with friction. The first of these explanations is that they are blown along with the prevailing winds. Tropical cyclones usually start out in the doldrums where the winds are few and uncertain. It is pointed out that the general circulation of the atmosphere has a westward component in lower latitudes and an eastward component in higher latitudes. These blow the cyclones first west then east, but how they work up to the northward is not made clear. It is hardly worth while to rebut by serious proof—which can be done—such an explanation.

The second is that cyclones are governed in their motions by the existence of some distant High or Low. The gradient may be almost imperceptible, but according to this theory, should a cyclone arise on a line between a High and a Low, no matter how far apart, it is considered that it ought to move towards the Low, *i. e.* after a fashion. But very often it does not do so, not even after a fashion.

We shall now attempt to analyze the motion of a cyclone taking into consideration the frictional forces developed. The writer had hoped that some one else with greater facilities and ability might have taken up the problem as indicated in the article mentioned above in the Monthly Weather Review, but the possibility of predicting the path of a cyclone, if not for its complete run, certainly for a few days ahead, does not seem to have aroused any interest. There was a time when the electrician did not need to be a mathematician, but that day has passed. So with the study of Meteorology; the meteorologist of the future will have to be a mathematician first. In the cyclone, the grandest phenomenon by far, the Chef d'Oevre of his specialty, he will find a complicated problem of forces which can only be unravelled by the calculus of mathematics.

A cyclone rotating about its axis experiences a certain amount of friction as its under surface passes over the surface of the Earth. If the initial momental energy were not added to, it would soon come to rest by reason of this resisting couple. Now this friction couple exerts a force contrary to the original momental energy about the axis of the Earth and is therefore continually decreasing it. The original moment of momentum about the Earth's axis is expressed by the quantity $Mk^2 \omega \sin \vartheta + MR^2 \cos^2 \vartheta \,.\, \Omega$. where ϑ is the latitude of the cyclone. Since we shall suppose that by added energy the moment $Mk^2 \omega$ is kept always constant throughout the period that we are studying, it follows that the retarding frictional couple will be continually reducing the term $MR^2 \cos^2 \vartheta\, \Omega$. This at first will affect chiefly the velocity Ω, and the cyclone will tend to lag further and further behind the position it would have occupied if there were no friction. In a certain sense it can be considered to be continually screwing itself backwards. Now if the cyclone is urged backwards along a parallel of latitude this motion is opposed by frictional forces, so that the moment of the force opposing the original momental energy about the axis of the Earth, will be reduced by this amount. If l is the moment of the frictional force about the axis of the cyclone, which we suppose to be kept constant, and $2a$ is the arm of this couple and f is the force it exerts on this arm, and if f_1 is the force urging the cyclone backwards, then $l = f \,.\, 2a = f_1 \,.\, R \cos \vartheta$.

Now if f_{11} is the frictional resistance opposing the sliding of the cyclone bodily, we have for the effective backward moment $l_1 = l - f_{11} R \cos \vartheta$.

We can write the equation therefore

$$MR^2 \cos^2 \vartheta\, \Omega + Mk^2 \omega \sin \vartheta = K - M \int_0^t l_1\, dt \qquad (1)$$

where K is the moment of momentum about the Earth's axis at the beginning of the period under consideration.

Now the gyroscope will in general be urged away from its parallel of latitude (in the great majority of cases, to the North, as already explained) and the normal effective deflecting force will be

$$\frac{Mk^2\,\omega\,.\,\Omega\,.\,\cos\vartheta}{R} - MR\cos\vartheta\,.\,\sin\vartheta\,.\,\Omega^2 - Mf_{11}\,. \tag{2}$$

We can consider that the last two forces are in the nature of constraints and the remaining effective force is given by the expression (2) so that the body moves northward under this force without friction.

The total moment of momentum given in (1) will not be diminished therefore by this northward motion of the cyclone.* The term $Mk^2\,\omega\sin\vartheta$ will be increased to be sure, but this will be made up by a loss in the first term. In other words the northward acceleration is at the expense of a certain amount of the momental energy of the cyclone about the Earth's axis.

We can write therefore

$$\frac{Mk^2\,\omega\,\Omega}{R}\,.\cos\vartheta - MR\cos\vartheta\,.\sin\vartheta\,\Omega^2 - Mf_{11} = M\frac{d^2\vartheta}{dt^2}\,. \tag{3}$$

It is usual to write $\frac{d\psi}{dt}$ instead of Ω, ψ being the azimuth at any instant of the plane passing through the axis of the cyclone and the axis of the Earth.

The two following equations express therefore the whole motion of the cyclone.

$$R^2\cos^2\vartheta\,\frac{d\psi}{dt} + k^2\,\omega\sin\vartheta = K - \int_0^t l_1 dt\,. \tag{4}$$

$$\frac{k^2\,\omega}{R}\cdot\frac{d\psi}{dt} - R\sin\vartheta\,.\left(\frac{d\psi}{dt}\right)^2 - f_{11}\sec\vartheta = \sec\vartheta\,\frac{d^2\vartheta}{dt^2} \tag{5}$$

f_{11} of course changes sign with the direction of the azimuthal motion of the cyclone with respect to the Earth's surface. If the cyclone moves to the eastward, $l_1 = l + f_{11}\,R\cos\vartheta$.

We shall not here discuss whether f_{11} is a constant, or a function of the velocity, or a more complicated function. That probably can be determined by observation. We may write (4)

$$R\cos\vartheta\,v_a + k^2\,\omega\sin\vartheta = K - \int_0^t (l - f_{11}\,R\cos\vartheta) \tag{6}$$

where v_a is the actual azimuthal velocity of a cyclone.

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*Since these northward forces act in a plane passing through the axis of the Earth, it is clear that they cannot influence the total moment of momentum about this axis.

In finding the position of the cyclone relatively to the Earth this actual azimuthal velocity will have to be compounded with that of the Earth's surface at the point where the cyclone happens to be. If this compounded velocity happens to be zero, as is usually the case at some point or points of the cyclone's path, this marks a point where the cyclone is changing from a relatively westward to a relatively eastward velocity or vice versa. In the ordinary parabolic paths of tropical hurricanes the point of recurvation is such a point.

The equations (4) and (5) are not in general susceptible of direct integration. But if we consider in Equa. (6) f_{11} to be a constant, and if the cyclone is in a low latitude, so that the term $k^2 \omega \sin \vartheta$ can be neglected without any great error, then we can easily use the equation. The values K and f_{11} will have to be determined by observations. For short periods l_1 might be considered constant, and there would then be only the two values K and l_1 to be determined.

Having determined an approximate value for $\frac{d\psi}{dt}$ at some time ahead, this may be substituted in Equa. (5) and assuming the most probable value for ϑ, the northward velocity $\frac{d\vartheta}{dt}$ might be determined to a certain degree of approximation.

Using an approximate formula $v_p^2 = K(\vartheta - \vartheta_0) - K'(\vartheta - \vartheta_0)^2$ where ϑ_0 is the latitude in which the cyclone originated, and v_p is the polar velocity, the writer has found a close agreement in the cyclones to which it was applied.

We have hitherto supposed that the cyclone was moving in a still and isobaric atmosphere. The chief and vastly preponderating forces which shape the path of the cyclone have already been considered. We shall now consider how far a deviation of the general atmosphere from the normal condition we have supposed can modify the shape of this path.

Let us suppose that the cyclone lies on a gradient between some general High and some general Low. We suppose here merely a gradient without any general motion of the atmosphere (wind) towards the Low. As far as the writer can gather from observations of vortices on the surface of water they are not influenced by ripples passing by them. Their shape is distorted, but unless a wave breaks over them, filling up the funnel, their motion continues as if the ripple had not traversed them. The effect of an existing gradient is merely to cause a slightly greater pressure on one side than the other. Now a cyclone rarely has the same pressure on all points of its rim. A slightly greater centrifugal force on one side will cause a greater barometrical height at this point and will tend to distort the shape of the cyclone and its stream lines. But no cyclone is a perfect circle. Our equivalent gyroscope is merely the dynamical equivalent of an oval or irregular mass of rotating air. To the extent then that a gradient may modify slightly the shape of a cyclone and therefore its equivalent gyroscope, it may alter the path of a cyclone, but this effect must be extremely slight.

The effect of the prevailing winds, as has been stated before, has been given as the cause of the motion. Now it is true that a cyclone imbedded in a mass of moving air would be carried along with it, but as soon as motion ceased to be parallel, *i. e.* as soon as the plane of the cyclone were turned, gyroscopic forces would be developed. Friction also would divert it from the general surrounding motion.

But a cyclone is a very thin, wide disc of air extending often over hundreds of miles. As it is moving with tremendous force, with its thin edge against the surrounding atmosphere, a wind on one side can have no great effect. True it may crumple up and distort the extreme exposed edge, but in a large cyclone the effect would be inappreciable. Certain it is that cyclones work constantly through the trade wind zones and their paths in general do not seem to be appreciably distorted. In the case of a tornado, which is an extreme form of cyclone, where the diameter may be only a few yards, though the height is considerable, there are no gyroscopic forces set up and its run is extremely limited. Such a whirl is governed simply by the motion of the surrounding air. We see this in a whirl of leaves on a gusty autumn day.

It remains to consider a few of the effects of the motion of a cyclone upon that of the Earth itself. We have seen that the friction couple tends to drive the cyclone backwards around the Earth's axis; but the reaction against this also tends to accelerate the Earth's rotation. Now since these accelerations are always in the same direction for every cyclone, both in the northern and southern hemispheres, the effect must be cumulative. It is possible to compute approximately this effect from the energies and frequencies of cyclones. It must be extremely small. Still as it is cumulative, it might amount to a fraction of a second in the length of the day in a century.

The frictional couple of a cyclone tends to cause the Earth to revolve about an axis directly under its centre, and would do so if the Earth were at rest. The combination of this tendency and the actual rotation, causes the Earth to revolve about some axis on the meridian of the cyclone and between the actual axis and the axis of the cyclone. Naturally the forces of the friction couple are so small compared with the energy of rotation of the Earth about its axis, that this deviation of the instantaneous axis from the normal axis must be very minute. As cyclones are pretty regularly distributed over the Earth, both in the northern and in the southern hemispheres, their effect in changing the direction of the Earth's axis will, in a long period of time, be apt to be self-eliminatory. The West Indies and the Philippines—the homes of the cyclone—are 180 degrees apart. Still it is possible that they are a factor, even though to a slight extent, in a small (not more than 100 feet) periodical variation which has been observed in latitudes.

On the Primitive Groups of Class Ten.

By W. A. Manning.

Let s_1, s_2, ... be a complete set of conjugate substitutions of prime order (p) in a primitive group G. They generate a transitive group. A certain number (λ) of these conjugates may generate an intransitive subgroup I_1. We wish to prove, before taking up the groups of class 10, that *a substitution can always be found in the series* s_1, s_2, ... *which connects the letters of any given set of intransitivity of* I_1 *with the letters of some other set.*

Suppose that the theorem is not true. Take a particular transitive set of $I_1 \equiv \{s_1, s_2, \ldots . s_\lambda\}$ in the letters a_1, a_2, ... Choose from the series s_1, ... a substitution $s_{\lambda+1}$ which connects a_1, ... with new letters, and which, of all the substitutions connecting the letters a_1, ... with new letters α, has the fewest new letters in the cycles with a_1, ... There may be a number of such substitutions. Select that one which connects the fewest new letters β with the letters b. Now consider $I_2 = \{I_1, s_{\lambda+1}\}$. If no substitution of the series s_1, ... connects the extended set a_1, ... with another transitive set of I_2, we take $s_{\lambda+2}$ as we did $s_{\lambda+1}$, and continue in this way until we have an intransitive subgroup I_{e-1} and a substitution S_e which connects the extended set a_1, ... with some other set of I_{e-1}. It is now essential to consider closely the substitution s_{e-1} by which we pass from I_{e-2} to I_{e-1}. Let the letters of the first set be a_1, a_2, ... a_k, and let the remaining letters of I_{e-2}, which may or may not form a single transitive set, be denoted by b_1, b_2, ...; we shall speak of the letters a and the letters b.

Letters α, new to I_{e-2}, are connected with letters a by s_{e-1}. Since s_{e-1} is of prime order, any power of s_{e-1} connects a's and α's. If s_{e-1} has two α's in any one cycle, x can be chosen so that in s_{e-1}^x these two new letters are adjacent. Then unless s_{e-1}^x replaces all the k letters a by α's, one of the generators of the group $s_{e-1}^{-x} I_{e-2} s_{e-1}^x$ will connect a's and α's and displace fewer α's than does s_{e-1}, contrary to hypothesis. Suppose that s_{e-1}^x replaces every a by an α, but has two α's in the same cycle with two a's, thus:

$$s_{e-1}^x = (a_1 \alpha_1 \ldots . a_2 \alpha_2 \ldots .) \ldots .$$

We can choose y so that

$$s_{e-1}^{xy} = (a_1\, a_2 \,\ldots\ldots\, a_1\, a_2 \ldots\ldots) \ldots\ldots$$

and proceed as before. It is now evident that if one cycle of s_{e-1} displaces $p-1$ letters a, all the cycles containing a letter a displace $p-1$ a's. In this case

$$s_{e-2} = (a_1\, a_1'\, a_2'' \ldots\ldots a_1^{(p-1)})\,(a_2\, a_2'\, a_2'' \ldots\ldots a_2^{(p-1)}) \ldots\ldots (a_k\, a_k' \ldots\ldots a_k^{(p-1)}) \ldots\ldots,$$

and all the substitutions of the series s_1, s_2, ..., not in I_{e-2}, and which connect a's and new letters, are of this form. Since G is a primitive group there is in it some substitution t which replaces an a by an a, and an a by some other letter w. Consider the group $t^{-1}\, I_{e-2} t$. One of its generators σ connects an a and a letter w. This w cannot be a letter b. Then

$$\sigma = (a'\ c'\ c'' \ldots\ldots c^{(p-1)}) \ldots\ldots$$

is of the same type as s_{e-1}, with the k letters a found in k different cycles. This gives

$$t = aa' \ldots\ldots a''c \ldots\ldots ba''' \ldots\ldots$$

Then $tI_{e-2}t^{-1}$ leaves at least one a fixed and has a generator, similar to s_1, which connects letters a with other letters. We can at once conclude that s_{e-1} has not two letters a in any cycle, and further that when $p=2$, s_{e-1} replaces an a by an a. We may write

$$s_e = a_\mu \beta \ldots\ldots,$$

where β is connected with the b's by s_{e-1}, and μ is arbitrary. If s_{e-1} has two letters a, as $a_\mu\, a_{\mu+1}$, adjacent in one of its cycles,

$$s_{e-1}^{-1}\, s_e\, s_{e-1} = a_{\mu+1}\, b \ldots\ldots$$

Or if s_{e-1} leaves an a fixed, we can make a_μ that letter and have

$$s_{e-1}^{-1}\, s_e\, s_{e-1} = a_\mu\, b \ldots\ldots$$

Hence the theorem as stated is true.

All the primitive groups of class 10 which contain a substitution of degree 10 and order 5 are known.[*] It remains to determine those which do not contain such a substitution. It is convenient to determine first the diedral groups of class 10 which are generated by two substitutions of the form

$$s_1 = a_1\, a_2 \,.\, b_1\, b_2 \,.\, c_1\, c_2 \,.\, d_1\, d_2 \,.\, e_1\, e_2 \,.$$

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[*] Transactions of the American Mathematical Society, Vol. 4 (1903), p. 351.

For brevity D shall be any diedral rotation group. Only one of these groups D is transitive. It is a primitive group of degree 11 and order 22, not contained in a larger primitive group of class 10 of the type we are considering. We neglect it entirely in what follows.

Let s_1 and s_2 be the two generators of a D. The product $s_1 s_2 = S$ is a positive substitution.

If the degree of S is less than the degree of $\{s_1, s_2\}$, s_1 and s_2 have one or more cycles in common. They cannot have three cycles in common for then S would be of degree 8 at most. If s_1 and s_2 have two cycles in common, we have the group

$$D_1 \equiv \{s_1 = a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2, \ \ s_2 = a_1 a_2 \,.\, b_1 b_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2\}.$$

If s_1 and s_2 have just one cycle in common, S is of degree 12, 13, ... 16. Neglecting the two letters common to s_1 and s_2 we have to do with a positive intransitive D of class 8 and degree 12, 13, ... or 16. S, if of degree 12, is a regular substitution of order 2, 3, or 6. There are two groups:

$$D_2 \equiv \{s_1, \ s_2 = a_1 a_2 \,.\, b_1 c_1 \,.\, b_2 c_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2\},$$
$$D_3 \equiv \{s_1, \ s_2 = a_1 a_2 \,.\, b_1 \beta_1 \,.\, c_1 \gamma_1 \,.\, d_1 \delta_1 \,.\, e_1 \varepsilon_1\}.$$

Corresponding to $S^6 = 1$, the D of class 8 cannot be constructed. A substitution of degree 13 contained in G is necessarily regular. If S is of degree 14 and regular, it is of order 7 and there is no group D. If S is not regular, it is of order 4 and $\{s_1, s_2\}$ has three octic constituents. These cannot be so arranged as to give, with the other two constituents, a D generated by two negative substitutions of degree 10. Again no isomorphism can be set up when S is of degree 15. When S is of degree 16 we have

$$D_4 \equiv \{s_1, \ s_2 = a_1 a_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2 \,.\, \delta_1 \delta_2\}.$$

We can now assume that S is of the same degree as $\{s_1, s_2\}$.

If S is of degree 12, it is regular and is of order 2, 3, or 6. We have at once

$$D_5 \equiv \{s_1, \ s_2 = a_1 b_1 \,.\, a_2 b_2 \,.\, c_1 d_1 \,.\, c_2 d_2 \,.\, \alpha_1 \alpha_2 \,.\,\};$$
$$D_6 \equiv \{s_1, \ s_2 = a_1 b_1 \,.\, a_2 c_1 \,.\, b_2 c_2 \,.\, d_1 \alpha_1 \,.\, e_1 \beta_1 \,.\,\}, \ S^3 = 1;$$
$$D_7 \equiv \{s_1, \ s_2 = a_1 b_1 \,.\, b_2 c_1 \,.\, d_1 e_1 \,.\, d_2 \alpha_1 \,.\, e_2 \alpha_2 \ \}, \ S^6 = 1;$$
$$D_8 \equiv \{s_1, \ s_2 = a_1 b_1 \,.\, a_2 \alpha_1 \,.\, b_2 \alpha_2 \,.\, c_1 d_1 \,.\, d_2 e_1 \ \}.$$

The product S cannot be of degree 13. If S is of degree 14, it must be non-regular, and is then of order 4. We have

$$D_9 \equiv \{s_1,\ s_2 = a_1 b_1 \,.\, c_1 \alpha_1 \,.\, c_2 \alpha_2 \,.\, d_1 \alpha_3 \,.\, d_2 \alpha_4\},\ \ S^4 = 1\,;$$
$$D_{10} \equiv \{s_1,\ s_2 = a_1 b_1 \,.\, c_1 d_1 \,.\, e_1 \alpha_1 \,.\, e_2 \alpha_2 \,.\, \alpha_3 \alpha_4\}.$$

If S is of degree 15:

$$D_{11} \equiv \{s_1,\ s_2 = a_1 \alpha_1 \,.\, b_1 \beta_1 \,.\, c_1 \gamma_1 \,.\, d_1 \delta_1 \,.\, e_1 \varepsilon_1\}.$$

If S is of degree 16:

$$D_{12} \equiv \{s_1,\ s_2 = a_1 b_1 \,.\, a_2 b_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2\}.$$

The degrees 17, 18, 19 give no groups D.

When S displaces 20 letters, we have,

$$D_{13} \equiv \{s_1,\ s_2 = \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2 \,.\, \delta_1 \delta_2 \,.\, \varepsilon_1 \varepsilon_2\}.$$

Note that D_7 and D_8 contain D_5 and D_6, and that D_9 and D_{10} contain D_2 and D_5.

The D_7, and hence D_8, which is the same group, can be thrown out very quickly. It has two transitive systems of 6 letters each. There is a substitution s_3 of the series $s_1, s_2, \ldots.$ which connects these two systems and brings in at most 4 new letters, one in a cycle,[*] so that $E \equiv \{s_1, s_2, s_3\}$ is a transitive group of degree not greater than 16. We can write s_3 so that it has a cycle new to s_1, is not commutative with it and has more than 4 letters new to s_1. Hence E is of degree 15 at most. If E is of degree 12, it is of order 24 and the only substitutions of degree 10 found in it are the 6 belonging to $\{s_1, s_2\}$. If E is of degree 13, its positive subgroup is of class 12, and contains just one subgroup of order 13, which must then be invariant in the group E. But s_1 cannot transform a group of order 13 into itself. If E is of degree 14, the positive subgroup is of class 12 and of order $14 \cdot 6$. By Sylow's theorem this subgroup has a characteristic cyclic subgroup of order 7, which is then invariant in E. Again s_1 cannot transform a group of order 7 into itself. If E is of degree 15, the positive subgroup is transitive of class 12, and contains no substitutions of degree 13 or 14; hence the subgroup leaving one letter fixed leaves three fixed. Now a group of order $15 \cdot 6$ has a single invariant subgroup of order 5. In E this is impossible. We can now strike D_7 and D_8 from our list.

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[*] Transactions, &c., l. c.

The next to go is the group D_3. The three substitutions of degree 10 in D_8 are

$$
\begin{aligned}
s_1 &= a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2 \,, \\
s_2 &= a_1 a_2 \,.\, b_1 \beta \,.\, c_1 \gamma \,.\, d_1 \delta \,.\, e_1 \varepsilon \,, \\
s_3 &= a_1 a_2 \,.\, b_2 \beta \,.\, c_2 \gamma \,.\, d_2 \delta \,.\, e_2 \varepsilon \,.
\end{aligned}
$$

According to the theorem established in the beginning of this article, there is a substitution s, similar to s_1, in the group G, which connects the set $a_1 a_2$ with another set of D_8. We write $s = a_1 b_1 \ldots .$ If s leaves a_2 fixed we have $s_3 s s_1 s s_3 = a_1 b_1 \,.\, a_2 y \ldots .$ Then putting $s = a_1 b_1 \,.\, a_2 y \ldots ,$ we see first that y cannot be new to s_1. For now that D_7 and D_8 are thrown out, a group of the type $\{s_1, s\}$ is not found in the list $D_1, \ldots, D_{18}$. For the same reason y must belong to s_2 and not to s_3. Hence $y = c_1$, d_1, or e_1. Without loss of generality we can say that $y = c_1$. Now $\{s_1, s\}$ and $\{s_2, s\}$ are both D_6. Hence from $\{s_1, s\}$

$$s = a_1 b_1 \,.\, a_2 c_1 \,.\, b_2 c_2 \ldots .$$

But $(b_2 c_2)$ is a cycle new to s_2 and on that account $\{s_2, s\}$ cannot be D_6. Hence D_8 can be dropped from our list.

D_6 is next in order of difficulty. The three substitutions of order 2 are

$$
\begin{aligned}
s_1 &= a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2 \,, \\
s_2 &= a_1 b_1 \,.\, a_2 c_1 \,.\, b_2 c_2 \,.\, d_1 \alpha_1 \,.\, e_1 \alpha_2 \,, \\
s_3 &= a_2 b_2 \,.\, a_1 c_2 \,.\, b_1 c_1 \,.\, d_2 \alpha_1 \,.\, e_2 \alpha_2 \,.
\end{aligned}
$$

There is a substitution s similar to s_1, which connects the set $a_1 \ldots$ with the set $d_1 d_2 \alpha_1$, and brings in at most 4 new letters, one to a cycle. Suppose first that $E \equiv \{D_6, s\}$ is transitive. If E is of degree 12, it has 4 subgroups of order 3 and is an imprimitive group isomorphic to the symmetric group of degree 4. In $(abcd)$ *all* we get a transitive representation of class 10 on 12 letters only by taking as a new element in the multiplication table of the group a subgroup 1, (ab). This substitution (ab) is contained in one subgroup of order 4, in two of order 6, in one of order 8, and in one of order 12. Hence by Marggraff's extension of a theorem[*] due to Jordan, this group E, if contained in a primitive group, is contained in a doubly transitive group of degree 13, of order $13 \cdot 12 \cdot 2$.

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[*] Marggraff, *Dissertation, Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades.* Giessen (1889). This point is made clear in a paper soon to appear in the Transactions of the American Mathematical Society under the title *On Multiply Transitive Groups*, by the present writer.

This G^{13} would by Sylow's theorem have to have an invariant subgroup of order 13. This is here impossible. It is clear that E cannot be of degree 13, 14, 15, or 16, if we keep in mind the reasoning in the case of D_7.

Now s does not connect more than two systems of D_6. By means of the conjoin of the regular constituent of D_6 we can write without loss of generality

$$s = a_1 \alpha_1 . \dots .$$

Looking over the non-Abelian groups in the list $D_1, \dots,$ we see that s can have 2, 4, or 5 letters new to s_1.

If s has just two letters new to s_1, $\{s_1, s\}$ is D_6 and

$$s = a_1 \alpha_1 . e_i x \dots (a_2)(e_j),$$

with all the 6 letters b, c, d displaced. We now arrive at a contradiction. If $e_i = e_1$, by $\{s_2, s\}$, $x \neq \varepsilon$, and by $\{s_3, s\}$, $x = \varepsilon$; there is a similar contradiction if $e_i = e_2$.

Next let s have just 4 letters new to s_1. Here

$$s = a_1 \alpha_1 . a_2 x_1 . \genfrac{}{}{0pt}{}{b}{c} x_2 . \genfrac{}{}{0pt}{}{b}{c} x_3 . \genfrac{}{}{0pt}{}{cd}{bd},$$

or

$$s = a_1 \alpha_1 . a_2 x_1 . e_1 x_2 . e_2 x_3 . cd.$$

Comparison with s_2 and s_3 shows that these forms for s have to be rejected because of the number of new letters.

If s has 5 letters new to s_1,

$$s = a_1 \alpha_1 . b x_1 . c x_2 . d x_3 . e x_4 .$$

Clearly $x_4 \neq \alpha_2$. Then neither $\{s_2, s\}$ nor $\{s_3, s\}$ is possible. This clears D_6 out of the way.

D_9, and with it D_{10}, goes out quickly. There are 4 substitutions in D_9 of order 2:

$$s_1 = a_1 a_2 . b_1 b_2 . c_1 c_2 . d_1 d_2 . e_1 e_2 , \qquad s_2 = a_1 b_1 . c_1 \alpha_1 . c_2 \alpha_2 . d_1 \alpha_3 . d_2 \alpha_4$$
$$s_3 = a_1 b_2 . a_2 b_1 . \alpha_1 \alpha_2 . \alpha_3 \alpha_4 . e_1 e_4 , \qquad s_4 = a_2 b_2 . c_2 \alpha_1 . c_1 \alpha_2 . d_2 \alpha_3 . d_1 \alpha_4 .$$

The substitution s may have two forms:

$$(1)\ s = e_1 a_1 \dots, \qquad (2)\ s = e_1 c_1 \dots .$$

Consider $s = e_1 a_1 \dots .$ The groups $\{s_1, s\}$ and $\{s_3, s\}$ are D_9 or D_{10}. In either case

$$s = a_1 e_1 \dots (a_2)(e_2)(b_2).$$

If $\{s_1, s\}$ is D_9, $\{s_3, s\}$ is also D_9, and $s = a_1 e_1 \,.\, c_1 \alpha' \,.\, c_2 \alpha'' \,.\, d_1 \alpha''' \,.\, d_2 \alpha^{\mathrm{IV}}$, with only one letter, e_1, new to s_2; $\{s, s_2\}$ is not in our list. If $\{s_1, s\}$ is D_{10}

$$s = a_1 e_1 \,.\, b_1 x \,\ldots\ldots,$$

where x is a c or d. But $\{s_3, s\}$ is also D_{10}, and x is an α. Consider the second case, $s = e_1 c_1 \ldots\ldots$ If $s s_1 = s_1 s$, we have uniquely:

$$s = c_1 e_1 \,.\, c_2 e_2 \,.\, a_1 a_2 \,.\, \alpha_3 \beta_1 \,.\, \alpha_4 \beta_2 .$$

The group $\{s_1, s_2, s\}$ has three sets of intransitivity. We now have, going one step further, either

$$(1)\ s' = e_1 d_1 \,.\, e_2 d_2 \,.\, b_1 b_2 \,\ldots\ldots,$$

an impossibility on account of $\{s, s'\}$, or

$$(2)\ s' = e_1 \beta_1 \,\ldots\ldots$$

Now s' must either add new letters to the set $a_1 \ldots\ldots$ or connect $d_1 \ldots\ldots$ and $\alpha_1 \ldots\ldots$ In neither case can $s' = e_1 \beta_1 \ldots\ldots$ be constructed. Hence D_9 and D_{10} can be dropped from our list.

Only one non-Abelian D is left. It is D_{11}. If a substitution s of the series $s_1, \ldots\ldots$ connects two cycles of another substitution s' of the series, s and s' are commutative.

We now take up D_2. It is generated by

$$s_1 = a_1 a_2 \,.\, b_1 b_2 \,.\, c_1 c_2 \,.\, d_1 d_2 \,.\, e_1 e_2 ,$$

and

$$s_2 = a_1 a_2 \,.\, b_1 c_1 \,.\, b_2 c_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 .$$

There are two substitutions, (1) $s = a_1 b_1 \ldots\ldots$, (2) $s = a_1 d_1 \ldots\ldots$ to be considered in extending D_2. Let $s = a_1 b_1 \ldots\ldots$ Neither $s s_1 = s_1 s$ nor $s s_2 = s_2 s$ is possible. If $s = a_1 d_1 \ldots\ldots$, then $s s_1 = s_1 s$. But $\{s_2, a_1 d_1 \,.\, a_2 d_2 \ldots\ldots\}$ is impossible. Hence G never includes D_2.

The generators of D_5 are s_1 and

$$s_2 = a_1 b_1 \,.\, a_2 b_2 \,.\, c_1 d_1 \,.\, c_2 d_2 \,.\, \alpha_1 \alpha_2 .$$

There are again only two substitutions s to be examined. The first,

$$s = a_1 \alpha_1 \,.\, b_1 \alpha_2 \,.\, c_1 \beta_1 \ldots\ldots,$$

we reject at once. But the second,

$$s = a_1 c_1 \,.\, a_2 c_2 \,.\, b_1 d_1 \,.\, b_2 d_2 \,.\, \beta_1 \beta_2 ,$$

requires that the next group $\{s_1, s_2, s, s'\}$ be considered. Now s' must be $a_1 \alpha_1 \ldots\ldots$, an impossibility. D_5 is rejected. We throw out D_1 also. In it

$$s_2 = a_1 a_2 \,.\, b_1 b_2 \,.\, \alpha_1 \alpha_2 \,.\, \beta_1 \beta_2 \,.\, \gamma_1 \gamma_2 .$$

Obviously we cannot have $s = a_1 c_1 \,.\, a_2 c_2 \ldots\ldots$ If

$$s = a_1 b_1 \,.\, a_2 b_2 \,.\, \delta_1 \delta_2 \,.\, \varepsilon_1 \varepsilon_2 \,.\, \zeta_1 \zeta_2 ,$$

we pass step by step by means of the substitutions

$$\begin{aligned} s' &= a_1 \delta_1 \,.\, b_1 \delta_2 \,.\, c_1 \alpha_1 \,.\, d_1 \beta_1 \,.\, e_1 \gamma_1 , \\ s'' &= a_1 \varepsilon_1 \,.\, b_1 \varepsilon_2 \,.\, c\alpha' \,.\, d\beta' \,.\, e\gamma' , \\ s''' &= a_1 \zeta_1 \,.\, b_1 \zeta_2 \,.\, c\alpha' \,.\, d\beta' \,.\, e\gamma' , \ (\alpha' = \alpha_1 \text{ or } \alpha_2, \text{ \&c.}) \end{aligned}$$

to an intransitive group $\{s_1, \ldots\ldots s'''\}$, with which we must stop. Then D_1 may be rejected.

It is now clear that D_4 also may be struck from our list.

Only D_{11}, D_{12}, and D_{13} are left.

Consider D_{11}. It leads to two primitive groups of class 10. The substitutions of order 2 in D_{11} are s_1,

$$\begin{aligned} s_2 &= a_1 a_3 \,.\, b_1 b_3 \,.\, c_1 c_3 \,.\, d_1 d_3 \,.\, e_1 e_3 , \\ s_3 &= a_2 a_3 \,.\, b_2 b_3 \,.\, c_2 c_3 \,.\, d_2 d_3 \,.\, e_2 e_3 . \end{aligned}$$

There must be in $s_1, \ldots\ldots$ a fourth substitution s_4 non-commutative with two of these three, non-commutative with s_1 and s_2, say. This follows from the fact that $\{s_1, \ldots\ldots\}$ is transitive. There are only two forms we need assume for s_4:

$$s_4 = a_1 a_4 \,.\, b_1 b_4 \,.\, c_1 c_4 \,.\, d_1 d_4 \,.\, e_1 e_4 , \tag{1}$$

$$s_4 = a_2 b_3 \,.\, a_3 b_2 \,.\, c_1 c_4 \,.\, d_1 d_4 \,.\, e_1 e_4 . \tag{2}$$

Consider (2). The group $E \equiv \{D_{11}, s_4\}$ has in all 6 substitutions of degree 10. The two not yet written down are:

$$\begin{aligned} s_5 &= a_1 b_3 \,.\, a_3 b_1 \,.\, c_2 c_4 \,.\, d_2 d_4 \,.\, e_2 e_4 , \\ s_6 &= a_2 b_1 \,.\, a_1 b_2 \,.\, c_3 c_4 \,.\, d_3 d_4 \,.\, e_3 e_4 . \end{aligned}$$

There is in E a set of intransitivity of 6 letters, $a_1 \ldots\ldots$, so that we may extend E by means of

$$s_7 = a_1 c_1 \,.\, a_2 c_2 \,.\, a_3 c_3 \,.\, d_4 d_5 \,.\, e_4 e_5 ,$$

or

$$a_1 c_3 \,.\, a_3 c_1 \,.\, b_2 c_4 \,.\, d_2 d_5 \,.\, e_2 e_5 .$$

These two substitutions are conjugate under

$$t = a_2 b_2 \,.\, c_1 c_3 \,.\, c_2 c_4 \,.\, d_1 d_3 \,.\, d_2 d_4 \,.\, e_1 e_3 \,.\, e_2 e_4 ,$$

which also transforms s_2 into s_3, s_1 into s_6, and s_3 into s_4. Hence only the first form of s_7 need be examined. Beside the substitutions already written down $E' = \{E, s_7\}$ has the following three substitutions similar to s_1:

$$s_8 = a_1 c_4 \,.\, b_3 c_2 \,.\, b_2 c_3 \,.\, d_1 d_5 \,.\, e_1 e_5 \,,$$
$$s_9 = b_1 c_1 \,.\, a_2 c_4 \,.\, b_1 c_3 \,.\, d_2 d_5 \,.\, e_2 e_5 \,,$$
$$s_{10} = b_2 c_1 \,.\, b_1 c_2 \,.\, a_3 c_4 \,.\, d_3 d_5 \,.\, e_3 e_5 :$$

In extending E', two substitutions s_{11} are to be considered. The one, $s_{11} = a_1 d_3 \ldots$ is impossible. The other

$$s_{11} = a_1 d_1 \,.\, a_2 d_2 \,.\, a_3 d_3 \,.\, c_4 d_5 \,.\, e_4 e_6 \,,$$

gives the group E'', in which the other substitutions of order 2 and class 10 are

$$s_{12} = a_1 d_4 \,.\, b_3 d_2 \,.\, b_2 d_3 \,.\, c_1 d_5 \,.\, e_1 e_6 \,,$$
$$s_{13} = a_2 d_4 \,.\, b_3 d_1 \,.\, b_1 d_3 \,.\, c_2 d_5 \,.\, e_2 e_6 \,,$$
$$s_{14} = a_3 d_4 \,.\, b_1 d_2 \,.\, b_2 d_1 \,.\, c_3 d_5 \,.\, e_3 e_6 \,,$$
$$s_{15} = c_1 d_1 \,.\, c_2 d_2 \,.\, c_3 d_3 \,.\, c_4 d_4 \,.\, e_5 e_6 \,.$$

Finally

$$s_{16} = a_1 e_1 \,.\, a_2 e_2 \,.\, a_3 e_3 \,.\, c_4 e_5 \,.\, d_4 e_6$$

is unique. The transitive group $\{s_1, s_2, s_4, s_7, s_{11}, s_{16}\}$ is primitive of degree 21, and is isomorphic to the symmetric group on 7 letters. The subgroup leaving one letter fixed is $\{E', s_{16} s_{11} s_{16} = d_1 e_1 \,.\, d_2 e_2 \,.\, d_3 e_3 \,.\, d_4 e_4 \,.\, d_5 e_5\}$. This subgroup has two sets of intransitivity. It is of order $2\,(5\,!)$. Hence the isomorphism between $\{E'', s_{16}\} \equiv E'''$ and $(abcdefg)$ *all* is simple.

This group E''' cannot be contained in a doubly transitive group, for then we should have a substitution

$$s = a_1 e_6 \,.\, e_1 d_4 \,.\, d_1 e_4 \,.\, b_1 - \,.\, c_1 - \,,$$

similar to s_1, which cannot exist at the same time as s_4. This also means that no group containing E''' can have other substitutions similar to s_1. Nor can E''' be invariant in a larger group of degree 21.

There remains the first form of s_4:

$$s_4 = a_1 a_4 \,.\, b_1 b_4 \,.\, c_1 c_4 \,.\, d_1 d_4 \,.\, e_1 e_4 :$$

Only one form for s_7 is possible:

$$s_7 = a_1 a_5 \,.\, b_1 b_5 \,.\, c_1 c_5 \,.\, d_1 d_5 \,.\, e_1 e_5 \,.$$

Continuing as before we get the unique set of generating operators, s_1, s_2, s_4, s_7, and

$$\begin{aligned} s_1' &= a_1 b_1 \,.\, a_2 b_2 \,.\, a_3 b_3 \,.\, a_4 b_4 \,.\, a_5 b_5 , \\ s_2' &= a_1 c_1 \,.\, a_2 c_2 \,.\, a_3 c_3 \,.\, a_4 c_4 \,.\, a_5 c_5 , \\ s_3' &= a_1 d_1 \,.\, a_2 d_2 \,.\, a_3 d_3 \,.\, a_4 d_4 \,.\, a_5 d_5 , \\ s_4' &= a_1 e_1 \,.\, a_2 e_2 \,.\, a_3 e_3 \,.\, a_4 e_4 \,.\, a_5 e_5 . \end{aligned}$$

These give an imprimitive group H of order $(5\,!)^2$, and of degree 5^2. It is not possible to form another substitution s similar to s_1. Hence H is invariant in any primitive group of class 10 containing it. Adjoining the substitution

$$t = a_2 b_1 \,.\, a_3 c_1 \,.\, a_4 d_1 \,.\, a_5 e_1 \,.\, b_3 c_2 \,.\, b_4 d_2 \,.\, b_5 e_2 \,.\, c_4 d_3 \,.\, c_5 d_3 \,.\, d_5 e_4 ,$$

we have a primitive group $G \equiv \{H, t\}$ of class 10, degree 25, and order $2\,(5\,!)^2$.

For all integral values of k greater than 2 there exists a primitive group G of degree k^2 and class $2\,k$ with an invariant subgroup H of the same type as H^{25}. Thus the imprimitive group H is generated by

$$\begin{aligned} s_2 &= (a_1 a_2)\,(b_1 b_2)\,(c_1 c_2) \ldots\ldots\ldots (j_1 j_2)\,(k_1 k_2), \\ s_3 &= (a_1 a_3)\,(b_1 b_3)\,(c_1 c_3) \ldots\ldots\ldots (j_1 j_3)\,(k_1 k_3), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_k &= (a_1 a_k)\,(b_1 b_k)\,(c_1 c_k) \ldots\ldots\ldots (j_1 j_k)\,(k_1 k_k), \\ \sigma_b &= (a_1 b_1)\,(a_2 b_2)\,(a_3 b_3) \ldots\ldots\ldots (a_j b_j)\,(a_k b_k), \\ \sigma_c &= (a_1 c_1)\,(a_2 c_2)\,(a_3 c_3) \ldots\ldots\ldots (a_j c_j)\,(a_k c_k), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sigma_k &= (a_1 k_1)\,(a_2 k_2)\,(a_3 k_3) \ldots\ldots\ldots (a_j k_j)\,(a_k k_k). \end{aligned}$$

If now we adjoin to H the substitution

$$\begin{aligned} t = (a_1)\,(a_2 b_1)\,(a_3 c_1)\,(a_4 d_1) \ldots (a_j j_1)\,(a_k k_1) \\ (b_2)\,(b_3 c_2)\,(b_4 d_2) \ldots (b_j j_2)\,(b_k k_2) \\ (c_3)\,(c_4 d_3) \ldots (c_j j_3)\,(c_k k_3) \\ \ldots\ldots\ldots\ldots\ldots \\ (j_j)\,(j_k k_j) \\ (k_k), \end{aligned}$$

the group $G \equiv \{H, t\}$ is primitive since the subgroup

$$\{s_2, s_3, \ldots\, s_j, \sigma_b, \sigma_c, \ldots\, \sigma_j, t\},$$

of order $2[(k-1)!]^2$ and omitting k_k, is maximal when k is greater than 2.

This group G is the only transitive group in which H is invariant.

If a second transitive group (G') exists in which H is invariant, there is a primitive group $G'' \equiv \{G, G'\}$ of order greater than $2(k!)^2$ in which H is invariant. Let G_1, H_1, ... be the subgroups of G, H, ... respectively, which leave one letter fixed. Since systems of imprimitivity of H can be chosen in only two ways, G'' is simply transitive.* Then G_1'' is intransitive, and since it includes G_1, the two sets of intransitivity of the latter are also the sets of intransitivity of G_1''. Now the larger set of intransitivity of G_1 is exactly that group of the system we are considering which corresponds to the value $k-1$ of k. Hence if the theorem is true for one value of k, it is true for the next succeeding value. But it certainly holds for $k=3$. Therefore the theorem is true in general.

The transitive group generated by the complete set of conjugates s_1, ... cannot be Abelian. Then our task is completed. There are three primitive groups of class 10 which do not contain a substitution of order 5 on 10 letters: the diedral G^{11}_{22}; the $G^{21}_{7!}$ isomorphic to the symmetric group on 7 letters; and the new group $G^{25}_{2(5!)^2}$. We recall the four groups found in a previous paper:† the metacyclic $\pm\, G^{11}_{11\cdot 10}$, the Mathieu $\pm\, G^{12}_{12\cdot 11\cdot 10}$, and their positive subgroups. Two of these groups, the G^{21} and the G^{25}, are omitted by JORDAN ‡ and one group is in his list for class 10 which does not belong there. All of these groups belong to infinite systems. The system to which G^{21} belongs is that of the symmetric-k group written on $\frac{k(k-1)}{2}$ letters. These groups are of class $2k-4$ (not $2k-2$ as JORDAN has it) and are primitive if k is greater than 4.

PARIS, 19 *Jan.*, 1905.

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* *On the Primitive Groups of Class Eight.* Bulletin of the American Mathematical Society (1904), 2nd Ser., v. 10, p. 286. An invariant imprimitive subgroup of a multiply transitive group is at least three-fold imprimitive.

† Transactions of the American Mathematical Society, l. c.

‡ Comptes Rendus. Vol. 75 (1872), p. 1754.

On Certain Unicursal Twisted Curves.

By Virgil Snyder.

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1. When the tangents to a twisted curve belong to a linear complex, the curve must be self dual. If it be of order n, the class of the osculating developable is also n. Let P be any point not on the curve c_n. From P can be drawn n osculating planes, the points of osculation being $P_1, P_2 \ldots P_n$. Since PP_i passes through P_i and lies in the polar plane of P_i as to the complex (osculating plane of c_n at P_i) it belongs to the complex, hence the polar plane of P contains $P_1 \ldots P_n$. *If through any point the osculating planes be drawn to a twisted curve contained in a linear complex, the points of osculation will lie in a plane passing through the given point.*

2. Let c_n be projected into a plane curve from P. The n osculating planes from P will cut the plane of the projected curve in inflexional tangents, and the points of inflexion will be in a straight line. *When a plane curve* $\phi_n = 0$ *is the projection of a twisted curve of the same order which belongs to a linear complex it must have n points of inflexion on a straight line.* This necessary condition is not sufficient, as $\phi_4 = 0$ exist having four collinear inflexions, and no singular points, but a twisted quartic of genus 3 does not exist.

For a unicursal c_n however, the condition is easily proved to be necessary and sufficient. Such a curve ϕ_n has $3(n-2)$ inflexions. The $2(n-3)$ inflexions of ϕ_n which are not included among the n collinear ones, are also inflexions on the twisted c_n. In general, the tangents at the points of a twisted curve at which the osculating planes have four point contact are ordinary, but in this case the stationary planes coincide in pairs, forming $2(n-3)$ points of linear inflexion.*

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* The theorem of § 1 was proved analytically for $n = 4$ by Appell, *Grunert's Archiv*, vol. 62, p. 175 (1878). The second theorem was proved analytically for unicursal curves and a particular projection by Picard in *Annales de l'École Normale* (1877).

A unicursal twisted curve of order n which belongs to a linear complex and has no point singularities has $2(n-3)$ *points at which the tangents have three point contact.* It has no stationary planes except those containing the inflexional tangents.

3. The points of c_n in the osculating planes form a symmetric $(n-3)$ correspondence which is in general not an involution. The branch points of the correspondence are the points of tangency of the inflexional tangents. Hence it follows that if the osculating plane at A cuts c_n again at B, then the osculating plane at B will contain the point A. The lines joining the points of osculation to each of the $n-3$ residual points of the curve in the plane are therefore principal secants. In particular, if $n=4$, we see that the quartic belonging to a linear complex has an infinite number of principal secants. The general unicursal quartic has three, and they meet in one point; in the present case no three can intersect in one point, for they all belong to a linear complex, and the polar plane of the point would have to cut the quartic in six points.

4. Two kinds of curves will now be discussed, one having a pair of maximum line singularities, the other having a pair of maximum point singularities.

Consider the curve

$$x:y:z:w=\lambda^n:n\lambda^{n-1}:n\lambda:1. \tag{1}$$

It lies on the R_2 whose equation is $n^2xw=yz$, and has $x=0$, $y=0$; $z=0$, $w=0$ for tangents having n — 1 points of contact, the former at $(0,0,0,1)$, the latter at $(1,0,0,0)$. These points correspond to the values $0, \infty$, of λ. The equation of the osculating plane at λ is

$$(n-2)(\lambda^n w-x)+\lambda^y-\lambda^{n-1}z=0. \tag{2}$$

The plane (2) will meet c_n in points μ defined by

$$(n-2)(\lambda^n-\mu^n)-n\lambda\mu(\lambda^{n-2}-\mu^{n-2})=0, \tag{3}$$

which has the factor $(\lambda-\mu)^3$. The residual factor is a homogeneous symmetric correspondence between λ and μ, which is therefore composite. When n is an even integer, $\lambda+\mu$ is also a factor, each of the $\frac{n}{2}-2$ remaining factors being of the form

$$(\lambda+\mu)^2+\alpha_i\lambda\mu. \qquad \left[i=1,2.\dots\frac{n}{2}-2.\right] \tag{4}$$

When n is odd, each of the $\frac{n-3}{2}$ factors is of this form. If the ruled surface be constructed by connecting the point λ to each of the $n-3$ points μ, it will break up into factors. The expression (4) equated to zero is also factorable, but each factor will define the same scroll. The scrolls arising from the different factors of the form (4) are entirely distinct.

5. When n is even, the lines joining each point λ to $-\lambda$ form a surface on which c_n is a simple curve. The points $\lambda, -\lambda$ form a quadratic involution having $0, \infty$ for double elements. The equation of the scroll is

$$x^{\frac{n}{2}-1} z^{\frac{n}{2}} = y^{\frac{n}{2}} \cdot w^{\frac{n}{2}-1}, \tag{5}$$

i. e. a scroll of order $n-1$, having $y=0$, $z=0$, the line joining the points of inflexion, for $\frac{n}{2}$-fold directrix, and $x=0$, $w=0$, the intersection of the stationary planes, for $\left(\frac{n}{2}-1\right)$-fold directrix. The two directrices are polar lines with regard to R_2 on which c_n lies, and also with regard to the complex to which it belongs. The lines $x=0$, $y=0$ and $z=0$, $w=0$ are each $\left(\frac{n}{2}-1\right)$ fold generators. There are no pinch points on $x=0$, $w=0$, and the only ones on $y=0$, $z=0$ are at $(1, 0, 0, 0)$, $(0, 0, 0, 1)$. The osculating planes at λ and at $-\lambda$ form with the planes through λ and the directrices of the scroll a harmonic pencil. A general plane section has an $\frac{n}{2}$-fold point on one directrix, and an $\left(\frac{n}{2}-1\right)$-fold point on the other directrix and on each inflexional tangent. The singularity on the latter lines has the defining terms $\zeta^{\frac{n}{2}} = x \cdot \xi^{\frac{n}{2}-1}$, hence it counts for $\frac{n}{2}-2$ cusps. There are no other point singularities, hence the class is n. Finally, it is evident that c_n is an asymptotic line on the surface, since a generator at λ always lies in the osculating plane of λ. It is the lowest order that a scroll of order $n-1$, contained in a linear congruence can have, when each generator is a bisecant of the curve.

All the asymptotic lines on the scroll (5) are of the same form as c_n, each having the same lines for $n-1$ point tangents at the same points. For $n=4$ we have the most general c_4 contained in a linear complex. Two orthogonal projections of a series of such curves, as asymptotic lines on a cubic scroll, are drawn in the Bulletin of the American Math. Soc. vol. 5 (1899), p. 349.

6. The other scrolls are all of the same nature, all being defined by (4). The two values of μ corresponding to a given value of λ are of the form $\varkappa\lambda$ and $\frac{\lambda}{\varkappa}$. If we consider a projective relation τ between λ, μ, having $0, \infty$ for selfcorresponding elements, the equations will be of this form. Associated with every point λ are two points $\tau\lambda$ and μ such that $\tau\mu = \lambda$, hence the two factors $(\mu - \varkappa\lambda)(\varkappa\mu - \lambda)$ will define the same scroll. Since two generators issue from each point λ of c_n, the curve is a double curve on the surface. If in (1) we write the equations of a line joining λ to $\varkappa\lambda$, the result will be a scroll of order $2(n-1)$. Moreover, in the present case both generators lie in a plane containing the tangent, hence c_n is tacnodal, and every point of it is a uniplanar point on the surface, instead of a biplanar point, as in case of an ordinary double curve. Moreover, all the generators lie in the osculating planes of c_n, hence the curve is an asymptotic line on the surface, and indeed for each of the two branches. From the equations of a generator it appears that the inflexional tangents are each $\frac{1}{2}(n-3)(n-2)$-fold lines on the surface. If $\varkappa$ is an $(n-1)$ root of unity, the $n-1$ points $\lambda, \varkappa\lambda, \varkappa^2\lambda \ldots$ are all collinear. The surface is now R_2.

7. Finally, if a generator be defined by the value t of λ, and another be defined by s, the condition that the generators s and t intersect is a symmetric correspondence between s, t. Let $s+t=\sigma$ and $st=\tau$. The equation $F(\sigma, \tau)=0$ becomes in our case factorable, one factor of the form (4) being squared, and $2n-4$ other factors of the form $\sigma^2+\beta_i\tau$. From the theorem recently proved by Mr. Sisam it follows that each nodal factor is a unicursal curve of order n, having each generator as a bisecant, and the points of intersection define a projectivity having $0, \infty$ for selfcorresponding elements. Hence every component nodal curve passes through both of the points of inflexion.

8. The curve (1) will project into a plane curve having two tangents of $(n-1)$ points of contact, and n collinear points of inflexion. For points on the line joining $0, \infty$, the two former points coincide at a node. For $n=4$, the lemniscate belongs to this type.

9. The curve

$$x : y : z : w = \lambda^{2n+1} : \lambda^{n+1} : \lambda^{n-1} : 1 \tag{6}$$

has a similar property. It belongs to a linear complex and lies on a quadric surface. It has two n-fold points, all the tangents at each being coincident. The locus of the lines joining a point λ to the $2n-2$ points μ in the osculating plane breaks up into $n-1$ surfaces, each one rational, of order $2(n+1)$, and having the original curve for a tacnodal component of the nodal curve. The scroll has no multiple generators. The residual nodal curve on each component consists of $n-2$ other curves, each one rational, each of order $2n+1$, each cut by every generator twice. The points of intersection of a generator with each component determine a projectivity having the singular points $0, \infty$ for self-corresponding elements.

The condition for a tacnodal curve is also satisfied when $\mu = \alpha\lambda$, α being a $(n+1)$th root of unity. The points $\lambda, \alpha\lambda, \alpha^2\lambda, \ldots$ are now collinear; they are the points of intersection of the curve with the generators of the R_2 on which the curve lies. The points form an involution of order $n+1$, and R_2 is counted $(n+1)$ times. As in the preceding case, every point of (6) is a uniplanar point on each of the scrolls, and the curve is an asymptotic line for both branches. The simplest case for which these scrolls have a meaning is when $n=2$. The equations of a generator become

$$\lambda^3 z + 2\lambda^3 y - x = 0, \quad \lambda^3 w - 2\lambda z + y = 0,$$

and the equation of the scroll is

$$\begin{vmatrix} 2yw & 2z^2 & zy + xw \\ 2z^2 & 3yz - xw & 2y^2 \\ zy + xw & 2y^2 & 2xz \end{vmatrix} = 0. \tag{7}$$

An arbitrary plane will cut a sextic curve from this surface, having five tacnodes; a plane containing a generator will cut a quintic curve having three tacnodes and having the generator for bitangent. A plane containing two generators (osculating plane of c_5 at λ) will have a cusp at λ and a node at each of the residual points μ. The scroll (7) is not included in Wiman's list of sextic scrolls. He overlooked the entire possibility of such configurations. It is my number 50, but the equation is not derived.* A plane projection of (6) will be a rational c_{2n+1} with two points that are n-fold, all the tangents coinciding. It also has n^2 double points, and $2n+1$ points of inflexion, all collinear. When the center of projection is taken on the line joining the two cusps the apparent double points all coincide with the cusps, the tangents at which are in different directions. When the cusps are real, part of the points of inflexion must be imaginary. The simplest curve of this type is the nodal cubic. For c_5, compare types 143–145 of Field's enumeration of unicursal quintics.*

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* JOURNAL, vol. 27 (1905), p. 188.

10. The curve

$$x : y : z : w = \lambda^n : n\lambda^{n-1} : \binom{n}{2}\lambda^{n-2} : 1 \tag{8}$$

lies on an R_2, has an $(n-1)$-fold point at (0, 0, 0, 1) with coincident tangents, and a stationary plane having n-point contact at (0, 0, 0, 0), the plane being $w = 0$. The curve does not belong to a linear complex, but has some properties analogous to the preceding ones. The lines joining λ to the $n-3$ points μ generate $n-3$ different scrolls, each of order $n+1$, and of type $[2, n-1]$, *i. e.* it is generated by the line of intersection of a tangent plane to a quadric cone and that of a planar developable of class $n-1$, when the planes are in (1, 1) correspondence. The curve (8) is a double (not tacnodal) curve on each, but is an asymptotic line on one of the branches. When n is even, $n = 2m$, c_n lies on a cone $\varkappa_m$, $x^{m-1}w = z^m$, forming the complete intersection of $\varkappa_m$ and $\varkappa_2$, $y^2 = xz$. The nodal curve of each R_{n+1} is of the same form for each component scroll; it consists of $m-2$ distinct curves besides (8), each one having the generators for bisecants. A similar projectivity ($\mu = \varkappa\lambda$) is defined on each component double curve. The line joining λ to its harmonic conjugate with regard to $0, \infty$ (cusp, stationary plane) will generate $\varkappa_m$. The factor corresponding to it will always appear in the criterion for the factorability of the nodal curve. The simple factor defines another nodal curve which is cut but once by each generator. The order of this curve is m. When n is odd, the harmonic conjugate does not lie in the osculating plane of λ. For $n = 5$, the scrolls are of type 19 in my list (l. c. p. 177).

CORNELL UNIVERSITY, *September*, 1905.

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* P. Field, "On the Forms of Unicursal Quintic Curves," JOURNAL, vol. 26 (1904), p. 149.

Functions of Three Real Independent Variables.

By Henry Livingston Coar.

CHAPTER I.

Simultaneous Limits.

1. *Definitions.*—Let $u = f(x, y, z)$ be a function of the three real independent variables x, y, and z, and let u be completely defined and single valued for all values of the variables, that is to say, u shall have one and only one definite value for each set of values of the variables x, y, and z. If (x_0, y_0, z_0) is such a set of values of the variables, then these may represent the coordinates of a point in space of three dimensions. For such values (x_0, y_0, z_0) the function $f(x, y, z)$ will have a definite value which we shall denote by $f(x_0, y_0, z_0)$. It will be convenient to call $f(x_0, y_0, z_0)$ "the value of the function at the point (x_0, y_0, z_0)".

We must now define limiting values and limits. We say that x_0 is a limiting value of the variable x, if there exist values of x in every neighborhood of x_0. This may also be expressed by saying that the values of x are dense at x_0. Similarly, if x, y, and z are dense, respectively, at x_0, y_0, and z_0, then (x_0, y_0, z_0) is a limiting value of (x, y, z). In this case (x, y, z) may approach (x_0, y_0, z_0) in any one of three different ways, or, to use the language of geometry, the point (x, y, z) may approach the point (x_0, y_0, z_0) in any one of three different ways. First, we may let each of the variables vary by itself, that is, we may regard any two of the variables as constant and let the third vary. Secondly, we may let two of the variables vary simultaneously while we regard the third variable as constant. Lastly, we may let all three variables vary simultaneously. These three different kinds of variation of the variables give rise to three different kinds of limits.

Before defining these let us recall briefly the different kinds of limits we may have in the case of a function of two real variables, namely, single limits, double limits, and simultaneous limits. These are defined as follows, x_0 and y_0

being in every case the limiting values of x and y, respectively. We say that the single limit $\lim\limits_{x=x_0} f(x, y)$ exists and equals A, if having chosen an arbitrarily small $\sigma > 0$, we can then find a $\delta > 0$, such that $| f(x_0 + \delta, y_0) - A | < \sigma$. We say that the double limit $\lim\limits_{x=x_0} \lim\limits_{y=y_0} f(x, y)$ exists if each of the following limits exists: $\lim\limits_{y=y_0} f(x, y) = F(x, y_0)$ and $\lim\limits_{x=x_0} F(x, y_0) = F(x_0, y_0)$. Lastly we say that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y)$ exists and equals A, if having chosen an arbitrarily small $\sigma > 0$, we can then find a $\delta > 0$, such that $| f(x_0+\delta_1, y_0+\delta_2) - A | < \sigma$, for every pair of values (δ_1, δ_2), where δ_1 and δ_2 are independent of each other and satisfy simultaneously the inequalities $|\delta_1| =< \delta$, $|\delta_2| =< \delta$.

We shall now define the different kinds of limits for a function of three variables. In what follows the values x, y, and z are supposed to be dense at x_0, y_0, and z_0, respectively.

(1) We say that the limit $\lim\limits_{x=x_0} \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x, y, z)$ exists if the following single limits exist: $\lim\limits_{z=z_0} f(x, y, z) = F(x, y, z_0)$, $\lim\limits_{y=y_0} F(x, y, z_0) = F_1(x, y_0, z_0)$, and $\lim\limits_{z=z_0} F_1(x, y_0, z_0) = F_2(x_0, y_0, z_0)$. We call the limit $\lim\limits_{x=x_0} \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x, y, z)$ the *triple limit.*

(2) We say that the limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} \lim\limits_{z=z_0} f(x, y, z)$ exists, if the single limit $\lim\limits_{z=z_0} f(x, y, z) = F(x, y, z_0)$ and the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} F(x, y, z_0) = F_1(x_0, y_0, z_0)$, both exist. The limit $\lim\limits_{z=z_0} \lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y, z)$ is defined in an analogous manner. Each of these limits is called a *double simultaneous limit.*

(3) Finally we say that the limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals A, if having chosen at pleasure a $\sigma > 0$, we can then find a $\delta > 0$, such that

$$|f(x_0+\delta_1, y_0+\delta_2, z_0+\delta_3)-A| < \sigma$$

where δ_1, δ_2, and δ_3, are independent of each other and satisfy simultaneously the inequalities $|\delta_1| \leqq \delta$, $|\delta_2| \leqq \delta$, $|\delta_3| \leqq \delta$. The limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ is called *the simultaneous limit in three variables.*

In any discussion of limits we must distinguish clearly between the value of a function at a point (x_0, y_0, z_0) and the limiting value of the function as (x, y, z) approaches (x_0, y_0, z_0). The former is dependent simply upon the value of (x_0, y_0, z_0), while the latter depends only upon the values of the function in the neighborhood of the point (x_0, y_0, z_0) and not upon the value of the function at (x_0, y_0, z_0), as is apparent from the following example.

Example.—Let $f(x, y, z)=x^2+y^2+z^2$ for all points excepting the origin, where the value of the function shall be unity. Then $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z)=0$, while $f(0, 0, 0)=1$.

2. *Properties of simultaneous limits.*—It is evident that we may let (x, y, z) approach (x_0, y_0, z_0) in an infinite number of different ways, and, moreover, that we may let it do so simultaneously. We must therefore meet the question as to whether we shall always obtain the same limiting value of the function if we let the variable approach the limiting value in different ways. The following examples show that different approaches may give rise to different values of the limiting function. In every case we suppose that the approach is a continuous one.

Example 1. Let $f(x, y, z)=\dfrac{x^2-y^2-z^2}{x^2+y^2+z^2}$.

Now let $x=2t$, $y=3t$, $z=4t$, so that x, y, and z will approach zero simultaneously, if we let t approach zero. Then

$$f(x, y, z)=\frac{4t^2-9t^2-16t^2}{4t^2+9t^2+16t^2}=-\frac{21}{29},$$

so that in case of this approach $\lim\limits_{t=0} f(x, y, z) = -\frac{21}{29}$. Again let $x = t$, $y = -t$, and $z = 2t$, so that

$$f(x, y, z) = \frac{t^2 - t^2 - 4t^2}{t^2 + t^2 + 4t^2} = -\frac{2}{3},$$

so that $\lim\limits_{t=0} f(x, y, z) = -\frac{2}{3}$.

We see from this example that for the two different linear approaches in question we obtain two different values of the limiting function. In this case we can therefore evidently not say that the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x, y, z)$ exists. But even if we should obtain the same limiting value of the function for every simultaneous linear approach of (x, y, z) to (x_0, y_0, z_0), we cannot say that the simultaneous limit will exist, for it may be possible that we shall obtain other limiting values for approaches of higher order. This is illustrated by the following example.

Example 2. Let $f(x, y, z) = \frac{xy + z}{xy - z}$.

Now let $x = at$, $y = bt$, $z = ct$, where a, b, and c may have any values whatever, so that this will give us any linear approach. Then we have

$$f(x, y, z) = \frac{abt^2 + ct}{abt^2 - ct} = \frac{abt + c}{abt - c},$$

so that $\lim\limits_{t=0} f(x, y, z) = -1$, which is the limiting value for every linear approach.

Now let $x = t$, $y = t$, $z = t^2$, so that

$$f(x, y, z) = \frac{t^2 + t^2}{t^2 - t^2} = \infty,$$

so that this quadratic approach gives us an infinite value of the function.

The reason why we obtain different limiting values of the function in the above examples is to be found in the fact that the limiting value of the function depends only upon the values of the function in the neighborhood of the point $(0, 0, 0)$ and not upon the value of the function at the point $(0, 0, 0)$. We are now in a position to state and prove the following theorem.

Theorem 1.—The necessary and sufficient condition that the simultaneous limit $\lim\limits_{\substack{x=x_0\\y=y_0\\z=z_0}} f(x, y, z)$ shall exist is that we always obtain the same definite limiting value A of the function, no matter in what manner (x, y, z) approaches (x_0, y_0, z_0) simultaneously and continuously.

The condition is necessary, that is to say, if the simultaneous limit $\lim\limits_{\substack{x=x_0\\y=y_0\\z=z_0}} f(x, y, z)$ exists and equals A; then, no matter how x, y, and z approach (x_0, y_0, z_0) simultaneously and continuously, we always obtain the same limiting value of the function, namely A. Let $x = \phi(t)$, $y = \psi(t)$, and $z = \chi(t)$, where ϕ, ψ, and χ are continuous functions of t. Then as t approaches t_0, we have in the limit

$$\lim_{t=t_0} \phi(t) = \phi(t_0) = x_0, \; \lim_{t=t_0} \psi(t) = \psi(t_0) = y_0, \; \lim_{t=t_0} \chi(t) = \chi(t_0) = z_0.$$

Making these substitutions in the original function, we obtain

$$f(x, y, z) = f[\phi(t), \psi(t), \chi(t)].$$

Our conclusion asserts that $\lim\limits_{t=t_0} f[\phi(t), \psi(t), \chi(t)] = A$.

Suppose that this is not so. In that case we can choose at pleasure a $\sigma > 0$ and can then determine a $\bar{\delta} > 0$, such that

$$|f[\psi(t_0 + \delta), \psi(t_0 + \delta), \chi(t_0 + \delta) - A| > \sigma, \tag{1}$$

where $\delta < \bar{\delta}$. But since ϕ, ψ, and χ are continuous functions of t, we have

$$\begin{aligned}\phi(t_0 + \delta) &= \phi(t_0) + \delta_1 = x_0 + \delta_1,\\ \psi(t_0 + \delta) &= \psi(t_0) + \delta_2 = y_0 + \delta_2,\\ \chi(t_0 + \delta) &= \chi(t_0) + \delta_3 = z_0 + \delta_3,\end{aligned}$$

where $|\delta_1|, |\delta_2|$, and $|\delta_3|$ can be made as small as we please by making δ sufficiently small. We can therefore make each of the δ's less than $\bar{\delta}$ which we do. Substituting these values in (1), we obtain

$$|f(x_0 + \delta_1, y_0 + \delta_2, z_0 + \delta_3) - A| > \sigma.$$

This contradicts our definition of a simultaneous limit, since $|\delta_1| \leq \bar{\delta}, |\delta_2| \leq \bar{\delta}$, $|\delta_3| \leq \bar{\delta}$. Hence the condition is necessary.

The condition is also sufficient. In this case we have

$$\lim_{t = t_0} f[\phi(t), \psi(t), \chi(t)] = A,$$

and we must prove that

$$|f(x_0 + \delta_1, y_0 + \delta_2, z_0 + \delta_3) - A| < \sigma.$$

If this is not true we shall have

$$|f(x_0 + \delta_1, y_0 + \delta_2, z_0 + \delta_3) - A| \geqq \sigma,$$

and hence

$$|f[\psi(t_0 + \delta_1), \psi(t_0 + \delta_2), \chi(t_0 + \delta_3) - A| \geqq \sigma,$$

which contradicts the hypothesis.

It was not necessary to express everything in terms of the new parameter t. The substitution $z = F(x, y)$, continuous in (x, y) together, and $y = F_1(x)$, continuous in x, would have let us express everything in terms of x, and the nature of the proof would remain unchanged.

The theorem just proved is especially useful in enabling us to decide when the simultaneous limit does not exist, as is shown by examples 1 and 2 of this section.

Theorem 2.—If the simultaneous limit $\lim\limits_{\substack{x = x_0 \\ y = y_0 \\ z = z_0}} f(x, y, z)$ exists and equals A, then we must also have:

$$(a)\quad \lim_{\substack{x = x_0 \\ y = y_0}} f(x, y, z_0) = \lim_{\substack{x = x_0 \\ z = z_0}} f(x, y_0, z) = \lim_{\substack{y = y_0 \\ z = z_0}} f(x_0, y, z) = A.$$

$$(b)\quad \lim_{x = x_0} f(x, y_0, z_0) = \lim_{y = y_0} f(x_0, y, z_0) = \lim_{z = z_0} f(x_0, y_0, z) = A.$$

Case (*a*). Let $z = F(x, y) = z_0$. Then, since the simultaneous limit $\lim\limits_{\substack{x = x_0 \\ y = y_0 \\ z = z_0}} f(x, y, z)$ exists, we have (by theorem 1), when we select the definite approach $z = F(x, y) = z_0$,

$$|f[x, y, F(x, y)] - A| < \sigma$$

or

$$|f(x, y, z_0) - A| < \sigma,$$

which proves the truth of the statement. In a like manner we prove the theorem for the other limits.

Case (*b*). Let $y = \psi(t)$ and $z = \chi(t)$, where ψ and χ are continuous functions of t and where we also have

$$\lim_{t = t_0} \psi(t) = \psi(t_0) = y_0, \ \lim_{t = y_0} \chi(t) = \chi(t_0) = z_0.$$

This, therefore, gives us a definite manner of approach, and since the simultaneous limit exists by hypothesis, we have (by virtue of theorem 1) for this approach

$$| f[x, \psi(t_0), \chi(t_0)] - A | < \sigma,$$

or

$$| f(x, y_0, z_0) - A | < \sigma.$$

This proves the theorem. The proof for the other limits is similar.

The converse of theorem 2 is, however, not true, as is evident from the following examples.

Example 3. Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$.

In this example we have

$$\lim_{\substack{x = 0 \\ y = 0}} f(x,y,0) = \lim_{\substack{x = 0 \\ z = 0}} f(x,0,z) = \lim_{\substack{y = 0 \\ z = 0}} f(0,y,z) = 0,$$

but the simultaneous limit $\lim\limits_{\substack{x = 0 \\ y = 0 \\ z = 0}} f(x, y, z)$ does not exist; for let $y = z = mx$.

Then we have

$$\lim_{\substack{x=0 \\ y=0 \\ z=0}} \frac{xyz}{x^3+y^3+z^3} = \lim_{x=0} \frac{m^2x^3}{x^3+2m^3x^3} = \frac{m^2}{1+2m^3}$$

which may assume all values between 0 and 1/3, as m varies from 0 to 1.

Example 4. Let $f(x, y, z) = \dfrac{xyz^3}{x^3+y^3+z^6}$.

In this example we have

$$\lim_{x=0} f(x, 0, 0) = \lim_{y=0} f(0, y, 0) = \lim_{z=0} f(0, 0, z) = 0,$$

but the simultaneous limit $\lim\limits_{\substack{x=0 \\ y=0 \\ z=0}} f(x, y, z)$ does not exist; for let $y = z = mx$.

Then we have

$$\lim_{\substack{x=0\\y=0\\z=0}} \frac{xyz^2}{x^8+y^8+z^8} = \lim_{x=0} \frac{m^8x^4}{(1+m^8)x^8+m^6x^8} = 0$$

for this approach. Again, let $y=z=x^2$. This approach gives us

$$\lim_{\substack{x=0\\y=0\\z=0}} \frac{xyz^3}{x^8+y^8+z^6} = \lim_{x=0} \frac{x^8}{x^8+x^8+x^8} = 1/3.$$

The simultaneous limit, therefore, does not exist by virtue of theorem 1. In this example we obtain the same limiting value for every linear approach, but not for every other approach. This emphasizes the importance of examining all possible approaches when we wish to see if the simultaneous limit exists.

In theorem 2 we have shown that from the existence of the simultaneous limit in three variables follows the existence of certain single limits and also of certain simultaneous limits in two variables. We might, therefore, infer that in this case the following limits also exist:

(*a*) $\lim\limits_{x=x_0} \lim\limits_{y=y_0} f(x,y,z_0) = \lim\limits_{y=y_0} \lim\limits_{x=x_0} f(x,y,z_0) = \lim\limits_{x=x_0} \lim\limits_{z=z_0} f(x,y_0,z) =$

$\lim\limits_{z=z_0} \lim\limits_{x=x_0} f(x,y_0,z) = \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x_0,y,z) = \lim\limits_{z=z_0} \lim\limits_{y=y_0} f(x_0,y,z) = A.$

(*b*) $\lim\limits_{x=x_0} \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x,y,z) = \lim\limits_{x=x_0} \lim\limits_{z=z_0} \lim\limits_{y=y_0} f(x,y,z) = \text{etc.} = A.$

(*c*) $\lim\limits_{\substack{x=x_0\\y=y_0}} f(x,y,z_0+\varepsilon_1) = \lim\limits_{\substack{x=x_0\\z=z_0}} f(x,y_0+\varepsilon_2,z) = \lim\limits_{\substack{y=y_0\\z=z_0}} f(x_0+\varepsilon_3,y,z)$, where ε_1, ε_2 and ε_3 may be arbitrarily small. This inference is, however, not warranted, as is shown by the following examples.

Example 5. Let $f(x,y,z) = z \sin 1/xy$.

In this case we always obtain the same limiting value zero at the origin, no matter in what manner x, y, and z approach simultaneously to $(0, 0, 0)$; hence the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x, y, z)$ exists, by theorem 1. It follows from theorem 2 that

$$\lim_{\substack{x=0\\y=0}} f(x, y, 0) = \lim_{\substack{x=0\\z=0}} f(x, 0, z) = \lim_{\substack{y=0\\z=0}} f(0, y, z) = 0,$$

and also that

$$\lim_{x=0} f(x, 0, 0) = \lim_{y=0} f(0, y, 0) = \lim_{z=0} f(0, 0, z) = 0.$$

But the limits under (*a*) do not exist, for $\lim\limits_{x=0} \lim\limits_{y=0} z \sin 1/xy$ has no meaning.

The same example also shows that the limits under (*b*) need not exist, for $\lim\limits_{z=0} \lim\limits_{x=0} \lim\limits_{y=0} z \sin 1/xy$ has no meaning. The existence of this limit would namely imply the existence of the limit $\lim\limits_{x=0} \lim\limits_{y=0} z \sin 1/xy$. This, however, does not exist, for if we let $z=2$, then the limit $\lim\limits_{x=0} \lim\limits_{y=0} z \sin \dfrac{1}{xy}$ may assume any value between $+2$ and -2. This example therefore shows that the existence of the simultaneous limit in three variables is not sufficient for the existence of the limits under (*a*) and (*b*). In the following example none of the limits under (*c*) exist.

Example 6. Let $f(x, y, z) = x \sin 1/y \sin 1/z$.

Here the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x, y, z)$ exists, but the limit $\lim\limits_{y=0} \lim\limits_{z=0} \varepsilon \sin 1/y \sin 1/z$ has no meaning, since it may assume all values between $+\varepsilon$ and $-\varepsilon$.

The converse of theorem 2 is also not true, that is, the simultaneous limits $\lim\limits_{\substack{x=x_0\\y=y_0}} f(x, y, z)$, $\lim\limits_{\substack{y=y_0\\z=z_0}} f(x, y, z)$, etc., may exist for every x, y, and z, respectively, in the neighborhood of (x_0, y_0, z_0), without necessitating the existence of the simultaneous limit in three variables. Likewise the single limits $\lim\limits_{x=x_0} f(x, y, z)$, etc., may exist for every pair of value (x, y), (x, z), (y, z), in the neighborhood

(x_0, y_0, z_0), while, nevertheless, the simultaneous limit in three variables does not exist. We show this by means of the following example.

Example 7. Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$.

The following simultaneous limits exist:

$$\lim_{\substack{x=0\\ y=0}} f(x, y, z) = \lim_{\substack{x=0\\ y=0}} \frac{xyz}{x^3 + y^3 + z^3} = 0, \text{ when } z = 0,$$

$$\lim_{\substack{x=0\\ y=0}} f(x, y, z) = \lim_{\substack{x=0\\ y=0}} \frac{xyz}{x^3 + y^3} = 0, \text{ when } z \neq 0.$$

We see at once from the symmetry of the function that the other simultaneous limits in two variables also exist and equal zero, but nevertheless the simultaneous limit in three variables does not exist, as has already been shown. In the case of this example the single limits also exist and are all equal to each other, namely,

$$\lim_{x=0} f(x, y, z) = \lim_{x=0} \frac{0}{x^3} = 0, \text{ when } y = 0 \text{ and } z = 0,$$

$$\lim_{x=0} f(x, y, z) = \lim_{x=0} \frac{0}{x^3 + y^3} = 0, \text{ when } y = 0 \text{ and } z \neq 0,$$

$$\lim_{x=0} f(x, y, z) = \lim_{x=0} \frac{xyz}{x^3 + y^3 + z^3} = 0, \text{ when } y \neq 0 \text{ and } z \neq 0.$$

From the symmetry of the function we see that the other single limits also exist and equal zero, but the simultaneous limit in three variables does not exist.

In theorem 2 we have deduced the existence of certain limits from the existence of the simultaneous limit in three variables. The following theorem takes us a step further.

Theorem 3.—If the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals A, and if also

$\lim\limits_{\substack{x=x_0\\y=y_0}} f(x,y,z)$ exists for every z in the neighborhood of z_0,

$\lim\limits_{\substack{x=x_0\\z=z_0}} f(x,y,z)$ " " y " " y_0,

$\lim\limits_{\substack{y=y_0\\z=z_0}} f(x,y,z)$ " " x " " x_0,

then we have

$$\lim_{\substack{x=x_0\\y=y_0\\z=z_0}} f(x,y,z)=\lim_{x=x_0}\lim_{\substack{y=y_0\\z=z_0}} f(x,y,z)=\lim_{y=y_0}\lim_{\substack{x=x_0\\z=z_0}} f(x,y,z)=\lim_{z=z_0}\lim_{\substack{x=x_0\\y=y_0}} f(x,y,z)=A.$$

Since by hypothesis the simultaneous limit in two variables exists, we have $\lim\limits_{\substack{x=x_0\\y=y_0}} f(x,y,z)=F(x_0,y_0,z)$, where however $F(x_0,y_0,z)$ need not equal $f(x_0,y_0,z)$. Hence

$$|F(x_0,y_0,z)-f(x_0+\delta_1,y_0+\delta_2,z)|<\sigma, \tag{1}$$

for every z where $z_0<z\leqq z_0+\delta_3$ and $|\delta_1|\leqq|\delta|, |\delta_2|\leqq|\delta|, |\delta_3|\leqq|\delta|$ simultaneously. Since, also by hypothesis, the simultaneous limit in three variables exists and equals A, we have

$$|f(x_0+\delta'_1,y_0+\delta'_2,z)-A|<\sigma, \tag{2}$$

for every value of z where $z_0<z\leqq z_0+\delta'_3$, and $|\delta'_1|\leqq|\delta|, |\delta'_2|\leqq|\delta|, |\delta'_3|=|\delta|$ simultaneously. Now let $|\delta_1|\leqq|\delta'_1|, |\delta_2|\leqq|\delta'_2|, |\delta_3|\leqq|\delta'_3|$; then we have

$$|f(x_0+\delta_1,y_0+\delta_2,z)-A|<\sigma, \tag{3}$$

for every value of z, where $z_0<z\leqq z_0+\delta_3$. We may therefore add (1) and (3) and thus obtain

$$|F(x_0,y_0,z)-A|<2\sigma,$$

for every value of z, where $z_0<z\leqq z_0+\delta_3$. Hence

$$\lim_{z=z_0} F(x_0,y_0,z)=A;$$

and therefore

$$\lim_{z=z_0}\lim_{\substack{x=x_0\\y=y_0}} f(x,y,z)=A.$$

In a similar manner we can prove that each of the other limits in question equals A, and since all these limits are therefore equal to A, it follows that the theorem is true.

Theorem 4.—If the simultaneous limit in three variables exists and equals A, and if also

$$\left.\begin{array}{ll} \lim\limits_{x=x_0} f(x,y,z) \text{ exists for every pair of values } & (y,z) \\ \lim\limits_{y=y_0} f(x,y,z) \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} & (x,z) \\ \lim\limits_{z=z_0} f(x,y,z) \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} \quad \text{“} & (x,y) \end{array}\right\} \begin{array}{c}\text{in the neighbor-}\\ \text{hood of}\\ (x_0,y_0,z_0),\end{array}$$

then we may also interchange the order of limits and have

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x,y,z)=\lim_{z=z_0}\lim_{\substack{x=x_0\\ y=y_0}} f(x,y,z)=\lim_{\substack{x=x_0\\ y=y_0}}\lim_{z=z_0} f(x,y,z)=\text{etc.}=A.$$

The proof of this theorem is similar to that of theorem 3. Since by hypothesis the single limits exist, it follows that $\lim\limits_{z=z_0} f(x,y,z)=F(x,y,z_0)$, for every value of (x,y) where $x_0<x\leqq x_0+\delta_1$, and $y_0<y\leqq y_0+\delta_2$, and $|\delta_1|\leqq|\delta|$, $|\delta_2|\leqq|\delta|$ simultaneously. Hence

$$|F(x,y,z_0)-f(x,y,z_0+\delta_3)|<\sigma,\ |\delta_3|\leqq|\delta| \tag{1}$$

Since, furthermore, the simultaneous limit in three variables exists and equals A, we have

$$|f(x,y,z_0+\delta'_3)-A|<\sigma, \tag{2}$$

for the same σ and every value (x,y) where $x_0<x\leqq x_0+\delta'_1$, and $y_0<y\leqq y_0+\delta'_2$, and $|\delta'_1|=|\delta|$, $|\delta'_2|=|\delta|$, $|\delta'_3|=|\delta|$, simultaneously. Now let $|\delta'_1|=|\delta_1|$, $|\delta'_2|=|\delta_2|$, $|\delta'_3|=|\delta_3|$; then (1) and (2) will hold for the same δ's, and we may therefore add these inequalities and obtain

$$|F(x,y,z_0)-A|<2\sigma,$$

for every value (x,y) where $x_0<x\leqq x_0+\delta_1$, and $y_0<y\leqq y_0+\delta_2$, and where $\delta_1>0$ and $\delta_2>0$. Hence

$$\lim_{\substack{x=x_0\\ y=y_0}} F(x,y,z_0)=A;$$

and therefore,

$$\lim_{\substack{x=x_0\\ y=y_0}}\lim_{z=z_0} f(x,y,z)=A.$$

In a similar manner we establish the truth of the other limits and our theorem is proved.

3. *Regular Points. Oscillation of a Function at a Point.*—In the preceding sections we have been interested only in the question of the existence of the simultaneous limit in three variables at the point (x_0, y_0, z_0), but we have not said anything about the value of this limit. It was pointed out that the value of the limit depends only upon the value of the function in the neighborhood of the point and not upon the value $f(x_0, y_0, z_0)$ of the function at the point (x_0, y_0, z_0). Now it may happen that the value of the simultaneous limit equals the value of the function at the point, that is $\lim\limits_{\substack{x = x_0\\ y = y_0\\ z = z_0}} f(x, y, z) = f(x_0, y_0, z_0)$. In this case we may say that the function $f(x, y, z)$ converges regularly for the value (x_0, y_0, z_0), or, in the language of geometry, that (x_0, y_0, z_0) is a *regular point* of the function. It is evident that at such a point the function is continuous in all three variables taken together, for the condition $\lim\limits_{\substack{x = x_0\\ y = y_0\\ z = z_0}} f(x, y, z) = f(x_0, y_0, z_0)$ is exactly the condition that $f(x, y, z)$ is continuous in all three variables taken together. It follows at once from theorem 2 of section 2 that, if $f(x, y, z)$ is continuous in all three variables taken together, then it is also continuous in any two of the variables taken together, and is also continuous in each variable by itself. The converse of these statements is however not true, as is seen from examples 3 and 4 of the preceding section.

If the simultaneous limit in three variables does not exist at the point (x_0, y_0, z_0), then we call the point an *irregular point* of the function. For any continuous approach of (x, y, z) to such a point, one of two things must happen. In the first place we may obtain two different limiting values of the function for two different simultaneous approaches of (x, y, z) to (x_0, y_0, z_0). In this case we say that the function has a *discontinuity of the first kind** at (x_0, y_0, z_0). Examples of such functions have been given in the preceding section. In the second case we may not obtain a definite limiting value of the function at all for any simultaneous approach of (x, y, z) to (x_0, y_0, z_0). In this case we say that the function has a *discontinuity of the second kind** at the point (x_0, y_0, z_0). Thus the function $\sin 1/x + \sin 1/y + \sin 1/z$ has a discontinuity of the second kind at the origin. At such a point the function may be continuous with respect to each variable alone or with respect to any two of the variables taken together, though it will not be continuous with respect to all three taken together.

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* Cf. Dini: Grundlagen für die Theorie der Funktionen einer veränderlichen reellen Grösse; page 51.

Oscillation of a Function.—Let us construct a sphere of radius ρ about the point (x_0, y_0, z_0), and let us denote the upper and lower limits of the values of the function within this sphere by M and m, respectively. We then call the difference $M - m$ the oscillation of the function $f(x, y, z)$ within the sphere with respect to (x, y, z) taken together. The limit of the oscillation, as ρ decreases indefinitely, we define as the amount of the discontinuity of $f(x, y, z)$ in (x, y, z) together at the point (x_0, y_0, z_0). If, therefore, $M - m = k$, then $\lim\limits_{\rho = 0} k$ is the amount of the discontinuity of $f(x, y, z)$ in (x, y, z) together at (x_0, y_0, z_0), or simply the (x, y, z) discontinuity of $f(x, y, z)$ at (x_0, y_0, z_0). From the definition follows at once that at a regular point the limit of the oscillation, as ρ decreases indefinitely, is zero and conversely, if at (x_0, y_0, z_0) $\lim\limits_{\rho=0} k = 0$, then the point is a regular point. If, however, $\lim\limits_{\rho=0} k \neq 0$, then the point in question is an irregular point. Now we have just seen that we may have two kinds of irregular points. In future we shall always call $\lim\limits_{\rho=0} k$ the amount of the discontinuity in each of the two cases. The Germans call this "Sprung".

Example 1. Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$.

This function is continuous in each variable alone at the origin, but the limit of the oscillation in all three variables taken together is 1/3 at (0, 0, 0); hence the function is discontinuous with respect to (x, y, z) taken together at this point.

Example 2. Let $f(x, y, z) = \dfrac{xyz^2}{x^3 + y^3 + z^6}$, where $f(0, 0, 0) = 0$.

This function is continuous in each variable by itself at (0, 0, 0), but it has at this point an infinite discontinuity in all three variables taken together.

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*Cf. Dini: Grundlagen für die Theorie der Funktionen einer veränderlichen reellen Grösse; page 51.

4. *Interchange of the Order of Limits.*—In investigating certain questions, such as the interchange of the order of differentiation and integration, it is very necessary to know under what conditions we may interchange the order of sequence of the double and triple limits to be considered. In the case of a function of two variables it is only necessary to investigate under what conditions $\lim\limits_{x=x_0}\lim\limits_{y=y_0} f(x, y) = \lim\limits_{y=y_0}\lim\limits_{x=x_0} f(x, y)$. In the case of a function of three variables, however, there are six possible cases, but there is considerable similarity among these, so that we need not give an extended proof in every case. For the sake of convenience we shall always take the origin as the critical point, which we may do without loss of generality.

Case 1. The necessary and sufficient condition that

$$\lim_{x=0}\lim_{\substack{y=0\\z=0}} f(x, y, z) = \lim_{\substack{y=0\\z=0}}\lim_{x=0} f(x, y, z) = A$$

is that

(1) $\lim\limits_{\substack{y=0\\z=0}} f(x, y, z) = \phi(x)$ exists for every $x \neq 0$,

$\lim\limits_{x=0} f(x, y, z) = \psi(y, z)$ exists for every $y \neq 0$ and every $z \neq 0$;

(2) $\lim\limits_{\substack{y=0\\z=0}} \psi(y, z) = A$;

(3) that there shall exist a set of pairs of values (y_i, z_k), dense at $(y=0; z=0)$, and corresponding to each pair (y_i, z_k) for which the absolute value $|\sqrt{(y_i^2 + z_k^2)}| < r$, where $r > 0$ is some fixed quantity, there shall exist an interval $(0, \delta_{i,k})$ greater than zero, which may however vary with (y_i, z_k), such that for every value x of this interval we have

$$|f(x, y_i, z_k) - \phi(x)| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k},$$

and σ is an arbitrarily small positive number.

Considered geometrically we wish to determine the conditions under which the limit taken along PEO will equal the limit taken along PAO (see figure 1).

The geometrical interpretation of conditions (1) and (2) is at once apparent from the figure. Condition 3 asserts that corresponding to every point within a circle of radius r about the origin in the YZ-plane there must exist an interval

$\delta_{i,k} = O'R = OK$, such that the value of the function at any point of $O'R$ shall differ from the value of the function at any point of OK by as little as we please.

Proof. We shall first show that the condition stated is a necessary condition, hence our hypothesis is

$$\lim_{x=0} \lim_{\substack{y=0\\ z=0}} f(x, y, z) = \lim_{\substack{y=0\\ z=0}} \lim_{x=0} f(x, y, z) = A.$$

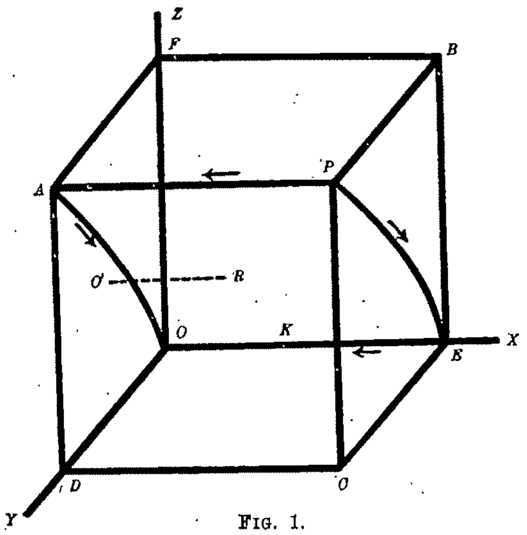

FIG. 1.

If this is so, then we have at once

$$(1) \qquad \lim_{\substack{y=0\\ z=0}} f(x, y, z) = \phi(x), \text{ when } x \neq 0,$$

$$\lim_{x=0} f(x, y, z) = \psi(y, z), \text{ when } y \neq 0 \text{ and } z \neq 0;$$

$$(2) \qquad \lim_{\substack{y=0\\ z=0}} \psi(y, z) = A, \text{ and } \lim_{x=0} \phi(x) = A;$$

so that the first two conditions of our theorem are fulfilled. Now, since $\lim_{x=0} \phi(x) = A$, we can choose an arbitrarily small $\sigma > 0$ and can then find a δ greater than zero, such that

$$|\phi(x) - A| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta. \qquad (a)$$

Since we also have $\lim\limits_{\substack{y=0\\z=0}} \psi(y, z) = A$, we can choose the same σ and can then determine a $\theta_1 > 0$ and a $\theta_2 > 0$, such that

$$|A - \psi(y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < y \leqq \theta_1 \text{ and } 0 < z \leqq \theta_2. \qquad (b)$$

But we have also $\lim\limits_{x=0} f(x, y_i, z_k) = \psi(y_i, z_k)$, hence with the same σ we can determine a $\delta_{i,k} > 0$, such that

$$|\psi(y_i, z_k) - f(x. y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta_{i,k}. \qquad (c)$$

Now let $\delta_{i,k}$ be the smaller of the two intervals δ and $\delta_{i,k}$, and add (a), (b), and (c). We then obtain

$$|\phi(x) - f(x, y_i, z_k)| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k},$$

and this holds for every y_i and z_k different from zero; hence it will hold for a set of pairs of values (y, z), dense at $y = 0$, $z = 0$, which proves our assertion.

We must now prove that the condition is also sufficient. We suppose therefore that the conditions (1), (2), and (3) of the theorem hold. From condition 2 follows at once that, having chosen an arbitrarily small $\sigma > 0$, we can determine a $\theta_1 > 0$ and a $\theta_2 > 0$, such that

$$|\psi(y_i, z_k) - A| < 1/3\,\sigma, \text{ where } 0 < y_i \leqq \theta_1, \text{ and } 0 < z_k \leqq \theta_2. \qquad (a)$$

From condition (3) follows that, having chosen the same σ, we can find a $\delta_{i,k} > 0$, such that

$$|\phi(x) - f(x, y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta_{i,k}, \qquad (b)$$

and where (y_i, z_k) is a pair of values of our set. From condition (1) we finally have that for the same σ, we can find a $\delta > 0$, such that

$$|f(x, y_i, z_k) - \psi(y_i, z_k)| < 1/3\,\sigma, \text{ where } 0 < x \leqq \delta. \qquad (c)$$

Now let $\delta_{i,k}$ be the smaller of the two quantities δ and $\delta_{i,k}$, and add (a), (b), and (c). We thus obtain

$$|\phi(x) - A| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k}.$$

Hence $\lim\limits_{x=0} \phi(x) = A$. But we have by hypothesis that $\phi(x) = \lim\limits_{\substack{y=0\\z=0}} f(x, y, z)$, hence $\lim\limits_{x=0} \lim\limits_{\substack{y=0\\z=0}} f(x, y, z) = A$, which proves the theorem.

In the above theorem no mention was made of the simultaneous limit in three variables; in fact, the theorem enables us to decide when we can interchange the order of sequence of limits even when the simultaneous limit in three variables does not exist.

Example 1. Let $f(x, y, z) = \frac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$.

Here $\lim\limits_{x=0} \lim\limits_{\substack{y=0 \\ z=0}} \frac{xyz}{x^3 + y^3 + z^3} = 0$ and $\lim\limits_{y=0} \lim\limits_{\substack{x=0 \\ z=0}} \frac{xyz}{x^3 + y^3 + z^3} = 0$, though the simultaneous limit $\lim\limits_{\substack{x=0 \\ y=0 \\ z=0}} f(x, y, z)$ does not exist.

Case 2. The necessary and sufficient condition that

$$\lim_{x=0} \lim_{\substack{y=0 \\ z=0}} f(x, y, z) = \lim_{y=0} \lim_{\substack{x=0 \\ z=0}} f(x, y, z) = A,$$

is that

(1) $\lim\limits_{\substack{y=0 \\ z=0}} f(x, y, z) = \phi(x)$ for every $x \neq 0$,

$\lim\limits_{\substack{x=0 \\ z=0}} f(x, y, z) = \psi(y)$ for every $y \neq 0$;

(2) $\lim\limits_{y=0} \psi(y) = A$;

(3) that we have a set of values $y_1, y_2, y_3, \ldots$ dense at $y = 0$, and corresponding to each y_n from some y on a region $(0 < x \leq \delta'_n,\ 0 < z \leq \delta''_n)$ of area different from zero, which may however vary with y_n, such that

$$|f(x, y_n, z) - \phi(x)| < \sigma, \text{ where } 0 < x \leq \delta'_n, \text{ and } 0 < z \leq \delta''_n.$$

Considered geometrically, we wish to determine the conditions under which the limit along *PEO* equals the limit along *PDO* (see figure 2). The geometrical interpretation of conditions (1) and (2) is at once evident. Condition (3) asserts that the value of the function at any point of *RSTU* shall differ from the value of the function at any point of *OG* by as little as we please.

Proof. The proof is similar to that of case (1). The condition is necessary; for if

$$\lim_{x=0}\ \lim_{\substack{y=0\\ z=0}} f(x,y,z)=\lim_{y=0}\ \lim_{\substack{x=0\\ z=0}} f(x,y,z)=A,$$

then surely conditions (1) and (2) must hold, and we shall have

$$\lim_{x=0}\ \lim_{\substack{y=0\\ z=0}} f(x,y,z)=\lim_{x=0}\ \phi(x)=A.$$

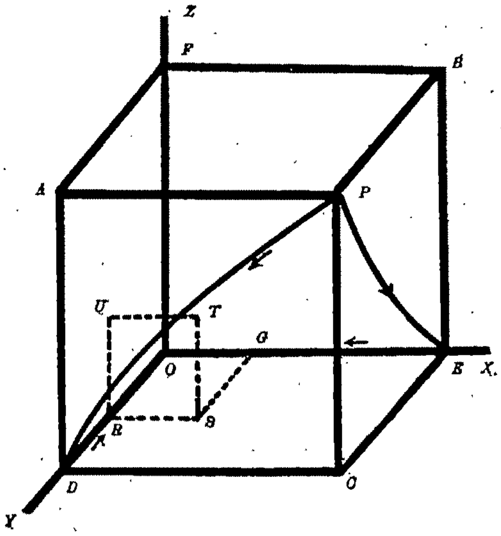

FIG. 2.

Hence we can choose at pleasure a positive number σ and can then find a $\delta>0$, such that

$$|\phi(x)-A|<1/3\,\sigma, \text{ where } 0<x\leqq\delta. \qquad (a)$$

Since $\lim\limits_{y=0}\psi(y)=A$, we can take the same σ and can then find a $\theta>0$, such that

$$|A-\psi(y_n)|<1/3\,\sigma, \text{ where } 0<y_n\leqq\theta, \qquad (b)$$

where y_n belongs to our set. Finally since $\lim\limits_{\substack{x=0\\ z=0}} f(x,y_n,z)=\psi(y_n)$, we can take the same σ and can then find a $\delta'_n>0$ and a $\delta''_n>0$, such that

$$|\psi(y_n)-f(x,y_n,z)|<1/3\,\sigma, \text{ where } 0<x\leqq\delta'_n, \text{ and } 0<z\leqq\delta''_n. \qquad (c)$$

Let δ'_n be the smaller of the two numbers δ and δ'_n. We can then add (a), (b), and (c), and obtain

$$|\phi(x) - f(x, y_n, z)| < \sigma, \text{ where } 0 < x \leqq \delta'_n \text{ and } 0 < z \leqq \delta''_n,$$

for every x and z within the region.

The condition is also sufficient; for we have from (2) that, having chosen an arbitrarily small $\sigma > 0$, we can then find a $\theta > 0$, such that

$$|\psi(y_n) - A| < 1/3\,\sigma, \text{ where } 0 < y_n \leqq \theta, \qquad (a)$$

where y_n belongs to our set. From (1) we have that with the same σ we can determine a δ' and a δ'', each greater than zero, such that

$$|f(x, y_n, z) - \psi(y_n)| < 1/3\,\sigma, \qquad (b)$$

where $0 < x \leqq \delta'$ and $0 < z \leqq \delta''$, and y_n is some number of our set. Finally we have from (3) that with the same σ we can find a $\delta'_n > 0$ and a $\delta''_n > 0$, such that

$$|\phi(x) - f(x, y_n, z)| < 1/3\,\sigma, \qquad (c)$$

where $0 < x \leqq \delta'_n$ and $0 < z \leqq \delta''_n$. Suppose now that $\delta'_n < \delta'$ and $\delta''_n < \delta''$. We can then add (a), (b), and (c), and thus obtain

$$|\phi(x) - A| < \sigma, \text{ where } 0 < x \leqq \delta'_n,$$

and hence it follows that $\lim\limits_{x=0} \phi(x) = A$. From this we have at once that $\lim\limits_{x=0} \lim\limits_{\substack{y=0\\z=0}} f(x, y, z) = A$, which proves the theorem.

Case 3. The necessary and sufficient condition that

$$\lim_{\substack{y=0\\z=0}} \lim_{x=0} f(x, y, z) = \lim_{\substack{x=0\\z=0}} \lim_{y=0} f(x, y, z) = A,$$

is that

(1) $\lim\limits_{y=0} f(x, y, z) = \phi(x, z)$, where $x \neq 0$ and $z \neq 0$.

$\lim\limits_{x=0} f(x, y, z) = \psi(y, z)$, where $y \neq 0$ and $z \neq 0$;

(2) $\lim\limits_{\substack{y=0\\x=0}} \psi(y, z) = A$;

(3) that we have a set of pairs of values (y_i, z_k) dense at $y = 0$, $z = 0$, and corresponding to each pair for which the absolute value $|\sqrt{(y_i^2 + z_k^2)}| < r$, where

r is some fixed positive number, an interval $\delta_{i,k}$ different from zero, which may however vary with (y_i, z_k), such that

$$| f(x, y_i, z_k) - \psi(x, z) | < \sigma,$$

where $0 < x \leqq \delta_{i,k}$, and $0 < z \leqq z_k$.

Considered geometrically we wish to determine the conditions under which the limit along PEO will equal the limit taken along PAO (see figure 3). The geometrical interpretation of conditions (1) and (2) is at once apparent from the figure. Condition (3) asserts that the value of the function at any point of $\delta_{i,k} = O'G$ shall differ by as little as we please from the value of the function at any point of $OHIK$, boundaries excluded. The proof in this case is exactly analogous to that of cases 1 and 2 and is therefore omitted.

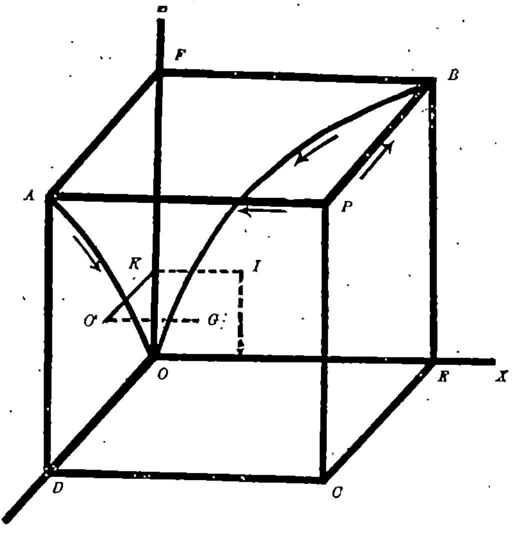

FIG. 3.

Case 4. The necessary and sufficient condition that

$$\lim_{y=0} \lim_{\substack{x=0\\ z=0}} f(x, y, z) = \lim_{y=0} \lim_{\substack{x=0\\ z=0}} f(x. y, z) = A,$$

is that

(1) $$\lim_{\substack{x=0\\ z=0}} f(x, y, z) = \phi(y), \text{ where } y \neq 0,$$

$$\lim_{x=0} f(x, y, z) = \psi(y, z), \text{ where } y \neq 0 \text{ and } z \neq 0;$$

(2) $$\lim_{\substack{y=0\\z=0}} \psi(y, z) = A;$$

(3) that there exists a set of pairs of values (y_i, z_k) dense at $y=0$, $z=0$, and corresponding to each pair of values for which $|\sqrt{(y_i^2+z_k^2)}|<r$, where r is some fixed positive number, an interval $(0, \delta_{i,k})$ different from zero, which may however vary with (y_i, z_k), such that

$$|\phi(y) - f(x, y_i, z_k)| < \sigma, \text{ where } 0 < x \leqq \delta_{i,k}.$$

Geometrically we are to determine the conditions under which the limit taken along PDO shall equal the limit taken along PAO (see figure 4). In this case the proof is exactly analogous to that of the three preceding cases and is therefore omitted.

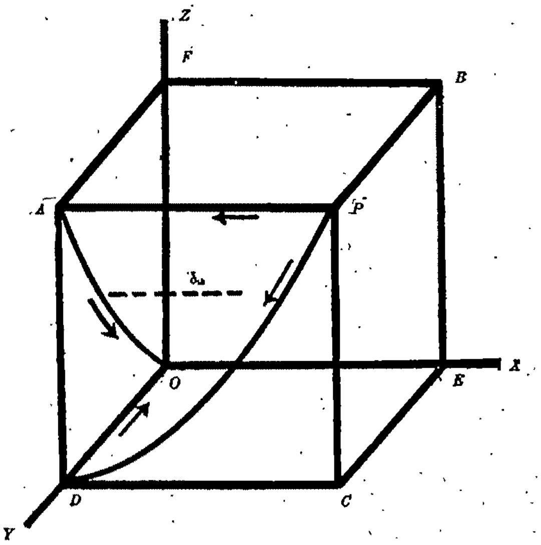

FIG. 4.

These four cases are those that involve both simple and simultaneous limits in two variables. We have still to consider the cases which involve only simple limits. Here we have two possibilities.

Case 5. The necessary and sufficient condition that

$$\lim_{x=0}\ \lim_{y=0}\ \lim_{z=0} f(x, y, z) = \lim_{z=0}\ \lim_{y=0}\ \lim_{x=0} f(x, y, z) = A,$$

is that

Now let δ_n be the smaller of δ_n and δ. Then we obtain by the addition of (a), (b), (c), and (d),

$$|\psi_1(x) - A| < \sigma, \text{ where } 0 < x \leqq \delta_n,\ 0 < y \leqq y_i,\ 0 < z \leqq z_k.$$

Hence $\lim\limits_{x=0} \psi_1(x) = A$, and therefore

$$\lim_{x=0}\ \lim_{y=0}\ \phi_1(x, y) = \lim_{x=0}\ \lim_{y=0}\ \lim_{z=0}\ f(x, y, z) = A,$$

which proves the theorem.

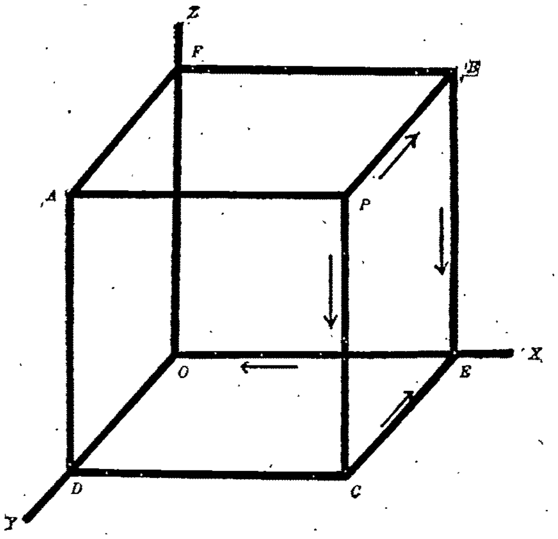

FIG. 6.

Case 6. This is the last possible case where we have to determine the conditions under which the limit taken along $PBEO$ equals the limit taken along $PCEO$ (see figure 6).

We are to find the conditions under which

$$\lim_{x=0}\ \lim_{y=0}\ \lim_{z=0}\ f(x, y, z) = \lim_{x=0}\ \lim_{z=0}\ \lim_{y=0}\ f(x, y, z) = A.$$

It is at once evident that this condition is the same as is the condition that

$$\lim_{y=0}\ \lim_{z=0}\ f(x_k, y, z) = \lim_{z=0}\ \lim_{y=0}\ f(x_k, y, z) \text{ for any } x_k.$$

It is therefore a question of simultaneous limits in two variables which we state.

The necessary and sufficient condition that

$$\lim_{y=0} \lim_{z=0} f(x, y, z) = \lim_{z=0} \lim_{y=0} f(x, y, z) = F(x),$$

for every x different from zero, is that for every such x we have

(1) $\lim\limits_{z=0} f(x, y, z) = \phi(x, y)$, where $y \neq 0$,

$\lim\limits_{y=0} f(x, y, z) = \psi(x, z)$, where $z \neq 0$;

(2) $\lim\limits_{z=0} \psi(x, z) = F(x)$;

(3) that for every such x there shall exist a set of numbers $y_1, y_2, y_3 \cdots$, dense at $y = 0$, and corresponding to each y of the set an interval $(0, \delta_i)$, such that

$$|\phi(x, y) - F(x)| < \sigma, \text{ where } 0 < z \leq \delta_i.$$

CHAPTER II.

PROPERTIES OF FUNCTIONS OF THREE VARIABLES.

5. *Continuity.*—In the case of a function of three variables we may have three different kinds of continuity.

(1) The function $f(x, y, z)$ may be continuous in each variable alone. That is to say, if we consider any two of the variables as constant, the resulting function of the third variable may be continuous in that variable alone.

(2) The function $f(x, y, z)$ may be continuous in two of the variables taken together. For example, if z is supposed to be constant, the resulting function of x and y may be continuous in (x, y) taken together. We say then that the function is continuous in two variables taken together, if the simultaneous limit $\lim\limits_{\substack{x=x_0 \\ y=y_0}} f(x, y, z)$ exists and equals the value of the function at the point in question.

(3) The function $f(x, y, z)$ may be continuous in all three variables taken together. We define this as follows: If about the point (x_0, y_0, z_0) we construct a sphere of radius ρ and denote the oscillation of the function within this sphere by k, and if we then have

$$\lim_{\rho=0} k = 0,$$

then we say that $f(x, y, z)$ is continuous in all three variables taken together at the point (x_0, y_0, z_0). As we have already seen in section 3, the point (x_0, y_0, z_0) is in this case a regular point of the function. We may therefore also say that $f(x, y, z)$ is continuous in all three variables taken together at (x_0, y_0, z_0), if the simultaneous limit in three variables $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$.

Points at which the limit of the oscillation, as ρ approaches zero, is different from zero have already been defined as irregular points. We may also call them points of discontinuity. The value of the limit of the oscillation k as ρ approaches zero we shall call the amount of the discontinuity. It is at once evident that if the function is discontinuous in any variable alone or in any two variables taken together, then it cannot be continuous in all three variables taken together. It is also evident that if the function is continuous in each variable alone, or if it is continuous in any two variables taken together, then it does not follow that it will be continuous in all three variables taken together. This is shown in some of the following examples.

Example 1. Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$.

At the origin this function is continuous in each variable alone and also in any two of the variables taken together, but it is not continuous in all three variables taken together, for the simultaneous limit $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z)$ does not exist.

Example 2. Let $f(x, y, z) = \dfrac{xy}{z}$, where $f(0, 0, 0) = 0$.

At the origin this function is not continuous in (x, y) taken together, but it is continuous in z alone, since when $x = 0$ and $y = 0$ we have $\lim\limits_{z=0} xy/z = 0$. But when $x \neq 0$ and $y = 0$, then $\lim\limits_{z=0} xy/z$ becomes infinite. Hence the function is not continuous at the origin in all three variables taken together.

Example 3. Let $f(x, y, z) = \dfrac{xy + z}{xy - z}$, where $f(0, 0, 0) = 1$.

At the origin this function is continuous in (x, y) together, for when $z=0$, $\lim\limits_{\substack{x=0\\y=0}} f(x,y,z)=1=f(0,0,0)$. When, however, $z\neq 0$, then $\lim\limits_{\substack{x=0\\y=0}} f(x,y,z)=-1$. Hence the function is not continuous in z alone and it cannot therefore be continuous at the origin in all three variables taken together.

Example 4. Let $f(x,y,z)=\dfrac{xyz}{xy+z^2}$, where $f(0,0,0)=0$.

This function is continuous at the origin in all three variables taken together, for the simultaneous limit $\lim\limits_{\substack{x=0\\y=0\\z=0}} f(x,y,z)=0$.

In all of the above examples we have dealt only with the continuity of a function at a point. In the following section we shall study the question of the continuity of a function in all three variables taken together throughout an entire region.

6. *Continuity of a Function Throughout a Region.*—In this and the next section we take up theorems that have to do with functions which are continuous throughout a region. We shall always assume that the function is singly determinate throughout the region, but for the present at least we shall not consider the behavior of the function on the boundaries of the region. We, therefore, define our region as follows:

$$\alpha<x<\beta,\ a<y<b,\ z_0<z<z_n.$$

It is not always necessary to consider all values of z within the above region, but it is sufficient for many purposes to take into consideration only a set of z's which is enumerable and dense at $z=z_0$. Unless, however, something is said to the contrary, we shall assume that z takes on all values of the above interval. With this understanding we now proceed to prove the following theorems.

Theorem 1.—If the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone throughout the region above defined, and if (x_0, y_0, z_0) is any point of the region, then we have

$$\lim_{\substack{x=x_0\\y=y_0}}\ \lim_{z=z_0} f(x,y,z)=\lim_{z=z_0}\ \lim_{\substack{x=x_0\\y=y_0}} f(x,y,z)=\lim_{x=x_0}\ \lim_{\substack{y=y_0\\z=z_0}} f(x,y,z)=$$

$$\lim_{y=y_0}\ \lim_{\substack{x=x_0\\z=z_0}} f(x,y,z)=\lim_{\substack{y=y_0\\z=z_0}}\ \lim_{x=x_0} f(x,y,z)=\lim_{x=x_0}\ \lim_{\substack{y=y_0\\z=z_0}} f(x,y,z)=f(x_0,y_0,z_0).$$

Since $f(x, y, z)$ is continuous in (x, y) taken together, we can consider z as constant and shall then have

$$\lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = f(x_0, y_0, z),$$

and this will hold for every z of the interval $z_0 < z < z_\kappa$. But we have by hypothesis that $f(x_0, y_0, z)$ is continuous in z alone, when we consider z as variable, and hence

$$\lim_{z=z_0} f(x_0, y_0, z) = f(x_0, y_0, z_0).$$

This gives us

$$\lim_{z=z_0} \lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = \lim_{z=z_0} f(x_0, y_0, z) = f(x_0, y_0, z_0),$$

which proves our theorem for this case. The other cases are proved in a similar manner.

Theorem 2.—If the function $f(x, y, z)$ is continuous in each variable alone throughout the region above defined and if (x_0, y_0, z_0) is any point of the region, then we have $\lim\limits_{x=x_0} \lim\limits_{y=y_0} \lim\limits_{z=z_0} f(x, y, z) = \lim\limits_{x=x_0} \lim\limits_{z=z_0} \lim\limits_{y=y_0} f(x, y, z) =$ $\lim\limits_{y=y_0} \lim\limits_{x=x_0} \lim\limits_{z=z_0} f(x,y,z) = \lim\limits_{y=y_0} \lim\limits_{z=z_0} \lim\limits_{x=x_0} f(x,y,z) = \lim\limits_{z=z_0} \lim\limits_{x=x_0} \lim\limits_{y=y_0} f(x,y,z) =$ $\lim\limits_{z=z_0} \lim\limits_{y=y_0} \lim\limits_{x=x_0} f(x, y, z) = f(x_0, y_0, z_0)$.

The proof is similar to that of theorem 1. Let us first consider x and y as constants; then, since the resulting function of z is continuous in z, we have

$$\lim_{z=z_0} f(x, y, z) = f(x, y, z_0),$$

and this will hold for every x and y of the respective intervals. But we have also by hypothesis that $f(x, y, z_0)$ is continuous in y alone, and hence if we consider x as constant, we shall have

$$\lim_{y=y_0} f(x, y, z_0) = f(x, y_0, z_0),$$

and this will hold for every x of the interval $\alpha < x < \beta$. Finally, since $f(x, y_0, z_0)$ is continuous in x alone, we have

$$\lim_{x=x_0} f(x, y_0, z_0) = f(x_0, y_0, z_0),$$

which proves our theorem for this case. The other cases are proved in a like manner.

Theorem 3.—If the function $f(x, y, z)$ is, throughout the region $\alpha < x < \beta$, $a < y < b$, $z_0 < z < z_n$, continuous in x alone, y alone, and z alone, and if we have a set of values (x, y, z) everywhere dense, in at least a part of the given region, in such a manner that they shall be dense at (x_0, y_0, z_0) along the lines $x = x_0, y = y_0, z = z_0$, where (x_0, y_0, z_0) is one of the limiting points, and if we let (x, y, z) approach (x_0, y_0, z_0) through this set of values in such a manner that no two of the variables vary together, then we always obtain the limiting value $f(x_0, y_0, z_0)$. Moreover, if we choose a second set of values (x, y, z) which shall be everywhere dense in the same subregion, and dense at (x_0, y_0, z_0) along the lines $x = x_0$, $y = y_0$, $z = z_0$, and let (x, y, z) approach (x_0, y_0, z_0) through this second set of values, then we obtain the same limiting value $f(x_0, y_0, z_0)$ as before.

The proof of this theorem follows from the fact that the limiting value of the function $f(x, y, z)$ at the point (x_0, y_0, z_0) depends only upon those values of the function in the neighborhood of (x_0, y_0, z_0) through which the function approaches its limiting value. Since now by hypothesis no two of the variables x, y, and z, are to vary together, the function $f(x, y, z)$ must eventually approach its limiting value in such way that two of the variables will be constant, for example $x = x_0$ and $y = y_0$. Then $f(x_0, y_0, z)$ is a continuous function of z, and hence we obtain the same limiting value $f(x_0, y_0, z_0)$, no matter through what set of points z approaches z_0. This proves our assertion. In a like manner we can prove it for the other cases.

7. *Continuity of a Function.*—(*Continued.*)—In this section we shall consider the number and the nature of the points within a given region, at which a function of three variables is continuous or discontinuous in all three variables taken together. As before, we shall not consider the behavior of the function on the boundaries of the region.

Theorem 1.[*]—If the points, at which the discontinuity of $f(x, y, z)$ in (x, y, z) taken together is greater than or equal to a given positive number σ, become dense at (x_0, y_0, z_0), then $f(x, y, z)$ is discontinuous in (x, y, z) taken together, also at (x_0, y_0, z_0), and the (x, y, z) discontinuity at (x_0, y_0, z_0) is greater than or equal to σ.

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[*] Cf. Schoenfliess: Goettinger Nachrichten, 1899, page 187.

In order to prove this theorem we need the following facts. If about the point P as center we have a sphere of radius ρ, then, as before, we denote the oscillation of the function within this sphere by k and the limit of the oscillation as ρ approaches zero by k'. If now $\bar{k}$ is some positive number greater than k', then it is evident that we can find about P some sphere within which the oscillation k of the function is less than $\bar{k}$. On the other hand, if $\bar{k}'$ is some positive number less than k', then we cannot find any sphere about P, such that within this sphere the oscillation k of the function is less than $\bar{k}'$.

This being established, we can prove our theorem without any difficulty. Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, ... be a set of points dense at $P_0(x_0, y_0, z_0)$. Suppose now that the theorem is not true. If now we construct about P_0 as center a sphere of radius ρ, then the limit k' of the oscillation of the function as ρ approaches zero will be less than σ. We can now find a positive number σ' satisfying the relation $k' < \sigma' < \sigma$. From the facts given at the beginning of the proof it therefore follows that we can find some sphere of radius r about P_0, such that within this sphere the oscillation k of the function is everywhere less than σ'. This sphere will surely contain some points of the set P_1, P_2, P_3, ... since these points are dense at P_0. Let P_k be one of the points within this sphere. Then at P_k the limit k' of the oscillation of the function is less than σ', that is, it is also less than σ, which is contrary to the hypothesis. Hence we are led to a contradiction, and the theorem must be true.

We next define what we mean when we say that a function is *pointwise discontinuous*. In this we follow Schoenfliess.*

If a function $f(x, y, z)$ is discontinuous in (x, y, z) taken together at a finite or an infinite set of points, but if at the same time it is continuous in (x, y, z) taken together at a set of points that is everywhere dense in the region in question, then we say that the function is pointwise discontinuous in (x, y, z) taken together.

Theorem 2.*—The necessary and sufficient condition that a function $f(x, y, z)$ is pointwise discontinuous in (x, y, z) taken together is that the points at which the amount of the (x, y, z) discontinuity is greater than or equal to σ are not everywhere dense in any subregion.

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* Cf. Schoenfliess: Jahresbericht der Deutschen Mathematiker-Vereinigung. Band 8, (1900), page 128.

The condition is necessary; for if the points at which the limit k' of the oscillation is greater than or equal to σ are everywhere dense in any subregion, then the limit k' of the oscillation would be greater than or equal to σ also at the limiting points, hence this limit k' would be greater than or equal to σ at every point of the subregion, which is contrary to the hypothesis.

The condition is also sufficient; for let us choose a set of positive quantities

$$\sigma_0 > \sigma_1 > \sigma_2 > \dots > \sigma_n > \dots$$

where

$$\lim_{n=\infty} \sigma_n = 0.$$

Corresponding to each of these quantities σ_i choose a set of points P_i which shall not be everywhere dense in any subregion, and such that at each of the points of P_i the amount k' of the (x, y, z) discontinuity is greater than or equal to σ_i, where $i = 1, 2, 3, \dots, i, \dots$ We thus obtain a sequence of sets of points

$$P_0, P_1, P_2, P_3, \dots, P_i \dots$$

Each of these sets certainly contains all of the points of all of the preceding sets. Now, since by hypothesis the points P_i at which the amount k' of the (x, y, z) discontinuity is greater than or equal to σ_i are not everywhere dense in any subregion, it follows that the points at which $k' < \sigma_i$ must be everywhere dense in every subregion, that is to say, in the whole region. We wish to show that the points at which $k' = 0$, as n increases indefinitely, are dense everywhere in the region. In order to do this, let R_0 be a sphere lying entirely within our region. Since the points of P_0 are not everywhere dense in any subregion, it follows either that the sphere R_0 contains no points of P_0 or that there exists a sphere R_1 lying entirely within R_0 such that it contains no points of P_0. Again the sphere R_1 either contains no points of P_1 or there exists a sphere R_2 lying wholly within R_1 which contains no points of P_1. Repeat this reasoning. We thus obtain a set of spheres $R_0, R_1, R_2, \dots, R_n$, each lying wholly within all of the preceding spheres, and such that the sphere R_n contains no points of the set P_{n-1}. If now we let n increase indefinitely, we obtain a limiting point P at which $k' = 0$, and hence at this point the function $f(x, y, z)$ is continuous in (x, y, z) taken together. It follows from this that within our region we can find no subregion which is free from points at which $f(x, y, z)$ is continuous in all three variables taken together. Hence our function can be at most pointwise discontinuous in all three variables together.

Theorem 3.—If throughout the region $\alpha < x < \beta$, $a < y < b$, $z_0 < z < z_n$, the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable by itself, then $f(x, y, z)$ can be at most pointwise discontinuous in all three variables taken together, that is to say, the points at which the (x, y, z) discontinuity is greater than or equal to some positive number $\bar{k}$ cannot be everywhere dense in any subregion.

Suppose that within our region there exists a subregion

$$\alpha' < x < \beta', \; a' < y < b', \; z'_0 < z < z'_n,$$

within which the (x, y, z) discontinuity is greater than or equal to $\bar{k}$ at a set of points everywhere dense. Then it follows from theorem 1 that the function

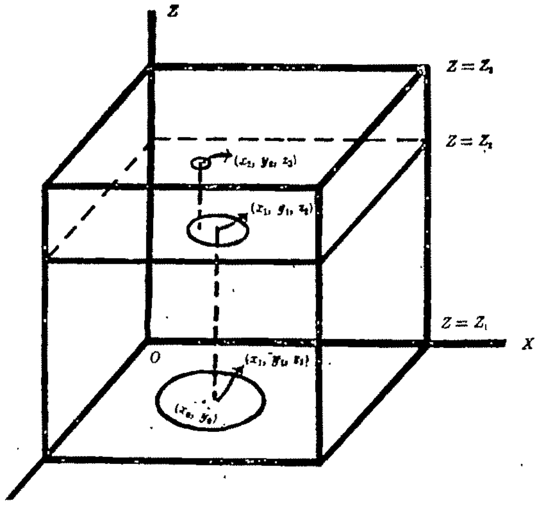

FIG. 7.

$f(x, y, z)$ must have an (x, y, z) discontinuity, greater than or equal to $\bar{k}$ at every point of this subregion. Let $P_0(x_0, y_0, z_1)$ be some point of this subregion and consider the plane $z = z_1$ parallel to the XY-plane (see figure 7). Since $f(x, y, z)$ is continuous in (x, y) together throughout the entire region, we can choose at pleasure a positive number σ and can then construct in the plane $z = z_1$ about P_0 a circle C_1 of radius r_1, such that for every point of this circle we have

$$|f(x, y, z_1) - f(x_0, y_0, z_1)| < \sigma. \qquad (1)$$

By hypothesis we have that at every point of the subregion the (x, y, z) discontinuity is greater than or equal to $\bar{k}$ so that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_1}} f(x, y, z)$ does not exist. We therefore can construct about P_0 as center a sphere R_1 of radius ρ_1 which we may choose less than r_1, such that for some points of this sphere we have

$$|f(x, y, z)-f(x_0, y_0, z_1)|>1/3\,\bar{k}.$$

Let (x_1, y_1, z_2) be such a point of the sphere R_1, so that

$$|f(x_0, y_0, z_1)-f(x_1, y_1, z_2)|>1/3\,\bar{k}. \tag{2}$$

By hypothesis $f(x, y, z)$ is continuous in (x, y) together, also at every point of the plane $z=z_2$, as far as it lies within the subregion. We can therefore construct in this plane about (x_1, y_1) a circle C_2 of radius $r_2<r_1$, lying wholly within the circle C_1 of radius r_1, such that for every point (x, y) of this circle we have

$$|f(x_1, y_1, z_2)-f(x, y, z_2)|<\sigma. \tag{3}$$

If we choose σ small enough and combine (1), (2), and (3), we obtain

$$|f(x, y, z_1)-f(x, y, z_2)|\geqq 1/4\,\bar{k}, \tag{4}$$

and this will hold for every point of the circle C_2 of radius r_2 in the plane $z=z_2$. Now let $(\bar{x}_2, \bar{y}_2)$ be some point within the circle C_2 of radius r_2, in the plane $z=z_2$. Since the (x, y, z) discontinuity of the function $f(x, y, z)$ at the point $(\bar{x}_2, \bar{y}_2, z_1)$ is greater than or equal to $\bar{k}$, we can construct about $(\bar{x}_2, \bar{y}_2, z_1)$ a sphere R_2 of radius ρ_2 which we choose less than r_2, such that for some points of this circle we have

$$|f(x, y, z)-f(\bar{x}_2, \bar{y}_2, z_1)|>1/3\,\bar{k}.$$

Let (x_2, y_2, z_3) be such a point of the sphere R_2, so that

$$|f(x_2, y_2, z_3)-f(\bar{x}_2, \bar{y}_2, z_1)|>1/3\,\bar{k}. \tag{5}$$

But the function $f(x, y, z)$ is continuous in (x, y) together at every point of the plane $z=z_3$, as far as it lies within our region; hence we can construct a circle C_3 of radius $r_3<r_2$ about (x_2, y_2) in the plane $z=z_3$, which circle shall lie wholly within the circle C_2 of radius r_2, such that for every point of this circle

$$|f(x, y, z_3)-f(x_2, y_2, z_3)|<\sigma. \tag{6}$$

From the continuity of $f(x, y, z)$ in (x, y) together we also have

$$|f(\bar{x}_2, \bar{y}_2, z_1)-f(x, y, z_1)<\sigma, \tag{7}$$

for all points of the circle C_3 of radius r_3 in the plane $z=z_3$. If now we choose σ small enough, we obtain by combining (5), (6), and (7),

$$|f(x, y, z_1)-f(x, y, z_3)| \geqq 1/4\,\bar{k}. \tag{8}$$

Continuing this reasoning, we obtain in general

$$|f(x, y, z_1)-f(x, y, z_n)| \geqq 1/4\,\bar{k}, \tag{9}$$

where (x, y) is any point of the circle C_n of the radius r_n, this circle lying wholly within each of a set of circles of radii

$$r_1 > r_2 > r_3 \ldots\ldots > r_n > \ldots.$$

The centers of these circles therefore approach a definite limiting point, say $(\bar{x}_0, \bar{y}_0)$. This point lies within each of the circles. Now we have by hypothesis that $\lim\limits_{\substack{x=\bar{x}_0\\ y=\bar{y}_0}} f(x, y, z_0)=f(\bar{x}_0, \bar{y}_0, z_0)$. But from (9) we have

$$|f(\bar{x}_0, \bar{y}_0, z_n)-f(\bar{x}_0, \bar{y}_0, z_0)| \geqq 1/4\,\bar{k}, \tag{10}$$

where $z_1, z_2, z_3, \ldots\ldots, z_n, \ldots.$ have the limiting value z_0. Hence we have from (10)

$$\lim_{z=z_0} f(x, y, z) \neq f(x, y, z_0)$$

for $x=\bar{x}_0$ and $y=\bar{y}_0$, which is contrary to the hypothesis that

$$\lim_{z=z_0} f(x, y, z) = f(x, y, z_0)$$

for every constant x and y. The assumption we made is therefore false and the theorem is true.

In the preceding theorem we have considered only the (x, y, z) discontinuities greater than or equal to some positive number $\bar{k}$. Let us now remove this restriction and consider all (x, y, z) discontinuities. We then have the following theorem.

Theorem 4.—If the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone, then it may be discontinuous in all three variables taken together at a set of points that is everywhere dense, but the amount of the discontinuity cannot be greater than or equal to $\bar{k}$ at each of these points.

We shall prove the first part of this theorem by means of an example.

Example. Let $\phi(x, y, z)=\dfrac{xyz}{x^3+y^3+z^3}$, where $\phi(0, 0, 0)=0$ and

$$-1 < x \leqq 1, \quad -1 < y \leqq 1, \quad -1 < z \leqq 1.$$

It has already been shown that this function is continuous in each variable alone and also in any two variables taken together, but it is not continuous in all three variables taken together at the point $(0, 0, 0)$, where the amount of the (x, y, z) discontinuity is $1/3$. Now let us form

$$\phi_k(x, y, z) = \frac{\sin^2 k!\,\pi x \sin^2 k!\,\pi y \sin^2 k!\,\pi z}{\sin^6 k!\,\pi x + \sin^6 k!\,\pi y + \sin^6 k!\,\pi z}.$$

This function is also continuous in each variable alone and in any two of the variables taken together, but it is discontinuous in all three variables taken together. Now let $k = 1$, so that

$$\phi_1(x, y, z) = \frac{\sin^2 \pi x \sin^2 \pi y \sin^2 \pi z}{\sin^6 \pi x + \sin^6 \pi y + \sin^6 \pi z}.$$

This function is discontinuous in (x, y, z) taken together at the points for which x equal 0 or 1, y equal 0 or 1, z equal 0 or 1, that is to say at the points $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $(0, 1, 1)$, $(1, 0, 0)$, $(1, 0, 1)$, $(1, 1, 0)$, $(1, 1, 1)$; for at these points the simultaneous limit $\lim\limits_{\substack{x = x_0 \\ y = y_0 \\ z = z_0}} \phi_1(x, y, z)$ does not exist. At all other points $\phi_1(x, y, z)$ is continuous in all three variables taken together. In a like manner we see that $\phi_k(x, y, z)$ is discontinuous in all three variables taken together at all points for which

$$\begin{aligned} x &= 0,\ 1/k!,\ 2/k!,\ 3/k!,\ \ldots\ldots\ k!/k!, \\ y &= 0,\ 1/k!,\ 2/k!,\ 3/k!,\ \ldots\ldots\ k!/k!, \\ z &= 0,\ 1/k!,\ 2/k!,\ 3/k!,\ \ldots\ldots\ k!/k!, \end{aligned}$$

while it is continuous in (x, y, z) taken together at all other points. Now form

$$f(x, y, z) = \sum_{n=1}^{\infty} 1/n!\,\phi_n(x, y, z).$$

Since $\phi_n(x, y, z) = 1/3$ for all values of n, we have

$$\sum_{n=1}^{\infty} 1/n!\,\phi_n(x, y, z) = 1/3 \sum_{n=1}^{\infty} 1/n!,$$

and hence $f(x, y, z)$ converges. Now consider z as constant. Then $f(x, y, z)$, considered as a function of x, y, and n, converges uniformly by theorem 5 of section 10. In a similar manner $f(x, y, z)$ converges uniformly as a function of z and n if we consider x and y as constants. Hence $f(x, y, z)$ is continuous in (x, y) together and in z alone. But $f(x, y, z)$ is discontinuous in all three variables taken together in every subregion, for the general term $1/n!\,\phi_n(x, y, z)$ is discontinuous in (x, y, z) together at every point for which

$$x = p/q,\ y = p'/q',\ z = p''/q'',$$

where the p's and q's are positive integers, such that p and q are relatively prime to each other; likewise p' and q', and p'' and q''. In order that this shall be true, we need only choose n so large that $n!$ is divisible by q, q', and q''. Then $1/n!\,\phi_n(x, y, z)$ will have an (x, y, z) discontinuity at the point $x = p/q$, $y = p'/q'$, $z = p''/q''$, and hence $f(x, y, z)$ also has an (x, y, z) discontinuity at this point. This shows us that $f(x, y, z)$ has an (x, y, z) discontinuity at every rational point of our region, that is to say, at a set of points that is everywhere dense, while at the same time it is continuous in all three variables taken together at all other points. This proves the first part of the theorem.

The truth of the second part of the theorem follows at once from theorem 3. For if the function could have an (x, y, z) discontinuity greater than or equal to some positive number $\bar{k}$ at a set of points that is everywhere dense, then it would also have such a discontinuity at the limiting points, hence at every point of the region, which is impossible.

Corollary.—If a function $f(x, y, z)$ is continuous in each variable alone, then it may be discontinuous in all three variables taken together at a set of points that is everywhere dense, but the amount of the (x, y, z) discontinuity can in this case not be greater than or equal to some positive number $\bar{k}$ at each of these points.

Theorem 5.—If throughout a given region, $a < x < \beta$, $a < y < b$, $z_0 < z < z_n$, the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone, then it is continuous in all three variables taken together at a set of points that is at least everywhere dense in the region.

In order to prove this we choose a set of positive quantities

$$\bar{k}_1 > \bar{k}_2 > \bar{k}_3 > \ldots\ldots\ldots\ldots > \bar{k}_n > \ldots\ldots$$

where

$$\lim_{n=\infty} \bar{k} = 0.$$

Now consider that portion of our region which is confined within any sphere R lying entirely within the given region. Then it follows from theorem 4 that the points at which $f(x, y, z)$ has an (x, y, z) discontinuity greater than or equal to $\bar{k}_1$ cannot be everywhere dense in any part of this sphere. We can therefore find a sphere R_1 of radius r_1, lying wholly within the sphere R, such that within R_1 there are no points at which the (x, y, z) discontinuity of $f(x, y, z)$ is greater than or equal to $\bar{k}_1$. Again we can find a sphere R_2 of radius r_2, lying wholly within the sphere R_1, such that within the sphere R_2 there are no points at which the (x, y, z) discontinuity of the function is greater than or equal to $\bar{k}_2$. Repeating this reasoning we finally obtain a sphere R_n of radius r_n lying wholly within the sphere R_{n-1} and hence within every preceding sphere, such that within this sphere R_n there are no points at which the (x, y, z) discontinuity of the function is greater than or equal to $\bar{k}_n$. But we have by assumption that $\lim\limits_{n=\infty} r_n = 0$, and we see from the manner in which the spheres were obtained that the centers of the spheres approach a definite limiting point (x_0, y_0, z_0), at which therefore we have $k = 0$; that is to say, the function $f(x, y, z)$ is continuous in (x, y, z) taken together at this point. From the manner in which the spheres were chosen it follows that the points at which the limit of the (x, y, z) discontinuity is zero must exist in every part of our region, however small. They are therefore at least everywhere dense in the region.

We can, however, go a step further than this. Schoenfliess* shows namely that the set of points in question must have the "*Mächtigkeit*" of the continuum. To do this, choose at pleasure a set of positive quantities

$$k_1 > k_2 > k_3 > \ldots\ldots > k_n > \ldots,$$

where $\lim\limits_{n=\infty} k_n = 0$. Now let P_1 be a set of points within the region at which the (x, y, z) discontinuity of the function is greater than or equal to k. Similarly, let P_2 be a set of points within the region at which the (x, y, z) discontinuity is greater than or equal to k_2, etc. In general let P_n be a set of points of the region at which the (x, y, z) discontinuity of the function is greater than or equal to k_n. It follows now that P_n is a set of points having the following properties. Each set P_n is in no subregion everywhere dense; each set contains all the points of every preceding set; the set given by $\lim\limits_{n=\infty} P_n$ is a set that may be everywhere dense in the region. Such a set is called by Baire† a set of points of the *first category*. Baire also shows that the set of points complementary to such a set is not a set of the first category. He calls it a set of the *second category* and proves that it has the "*Mächtigkeit*" of the continuum. Hence it follows that the points at which the (x, y, z) discontinuity of our function is greater than or equal to some positive number k' is a set of the first category. The set of points therefore for which $\lim\limits_{n=\infty} k_n = 0$ is the complementary set of the above set and hence is a set of the second category. It therefore has the "*Mächtigkeit*" of the continuum and is non-enumerable.

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* Cf. Schoenfliess: loco citato, page 129.

† Cf. Baire: Annali di matematici (3), vol. 3, page 67 (1899).

8. *Uniform Convergence.*—When dealing with the behavior of a function $f(x, y, z)$ not only within a given region but also on the boundaries of the region, we meet with a new kind of convergence, which has been called uniform convergence in the case of a function of two real variables. It is defined as follows.

If a function $f(x, y)$ of two real variables is defined for all values of $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_0 + \delta$, and if $\lim\limits_{y=y_0} f(x, y) = f(x, y_0)$ for every x of the interval $\alpha \leqq x \leqq \beta$, then we say that $f(x, y)$ converges uniformly to $f(x, y_0)$ if, having chosen an arbitrarily small $\sigma > 0$, we can then find a $\delta > 0$, such that

$$|f(x, y) - f(x, y_0)| < \sigma$$

for every y of $y_0 < y \leqq y_0 + \delta$, and for every x of the interval $\alpha \leqq x \leqq \beta$. This is equivalent to saying that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0}} f(x, y)$ must exist for every x of the interval $\alpha \leqq x \leqq \beta$, and must equal the value $f(x_0, y_0)$.

When we come to a function of three real variables we have two cases to consider; for the function may converge uniformly either with respect to one of the variables or with respect to two of the variables.

In the former of these two cases we must have a function of three variables defined for every point of the region

$$\alpha \leqq x \leqq \beta,\ a \leqq y \leqq b,\ z_0 < z \leqq z_n,$$

and we must also have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) of the region. We then say that $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$, if the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ at every point (x_0, y_0) of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$. This definition may also be expressed otherwise as follows.

The function $f(x, y, z)$ shall be defined for every point of the given region and $\lim\limits_{z = z_0} f(x, y, z)$ shall equal $f(x, y, z_0)$ for every point of the region as defined for x and y. If now

1. We choose at pleasure a $\sigma > 0$ and can then find a $\delta > 0$ such that

$$|f(x, y, z) - f(x, y, z_0)| < \sigma$$

when $z_0 < z \leqq z_0 + \delta$, for every point (x, y) of $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, then we say that $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$ throughout the region, boundaries included. Two things are essential in this definition. First, the same δ must fit every point (x, y) of the region, and, secondly, the region must include its boundaries.

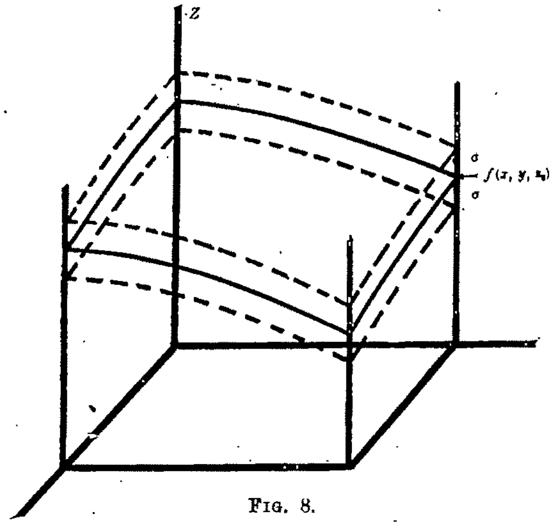

FIG. 8.

As in the case of a function of two variables, so here we have a geometrical interpretation of uniform convergence. The limiting function $f(x, y, z_0)$, considered as a function of x and y, may be represented by a surface. In a like manner we may represent the function $f(x, y, z_k)$, where $k = 1, 2, 3, \ldots\ldots$, by surfaces which we call the approximation surfaces. We now construct at a distance σ above and below $f(x, y, z_0)$ a surface parallel to $f(x, y, z_0)$, see figure 8.

If now we can find a z_k such that for every value of z, from z_k on, each of the approximation surfaces falls entirely within the strip bounded by the two parallel surfaces for every point (x, y) of the given region, then $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$.

In the second case, where we have uniform convergence with respect to two variables, the function $f(x, y, z)$ must be defined for every value (x, y, z) of the region

$$\alpha \leqq x \leqq \beta,\ y_0 < y \leqq y_m,\ z_0 < z \leqq z_n$$

and the limit $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z)$ must equal $f(x, y_0, z_0)$ for every x of the interval $\alpha \leqq x \leqq \beta$. If under these conditions the simultaneous limit $\lim\limits_{\substack{x = x_0 \\ y = y_0 \\ z = z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ for every value of x_0 of its interval, then $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$.

This can also be expressed otherwise. If $f(x, y, z)$ is defined throughout the given region and satisfies the condition that $\lim\limits_{\substack{x = x_0 \\ y = y_0 \\ z = z_0}} f(x, y, z) = f(x_0, y_0, z_0)$ for every x_0 of the interval $\alpha \leqq x \leqq \beta$, and if we choose an arbitrarily small $\sigma > 0$ and can then find a $\delta_1 > 0$ and a $\delta_2 > 0$, which are independent of each other and such that

$$|f(x, y, z) - f(x, y_0, z_0)| < \sigma$$

for every y and z where $y_0 < y \leqq y_0 + \delta_1$ and $z_0 < z \leqq z_0 + \delta_2$, then $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$ throughont the interval $\alpha \leqq x \leqq \beta$.

Geometrically we can represent $f(x, y_0, z_0)$ by a curve in the interval $\alpha \leqq x \leqq \beta$. Similarly, $f(x, y_i, z_k)$, where $i = 1, 2, 3, \ldots.$ and $k = 1, 2, 3, \ldots.$ can be represented by a set of curves, which we call the approximation curves. Now draw at a distance σ above and below $f(x, y_0, z_0)$ curves parallel to the projections of the approximation curves. If then there exists a (y_i, z_k) such that, for every (y, z) from (y_i, z_k) on, the projections of the approximation curves all fall in the strip bounded by the two parallel curves, then $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$.

The essential thing for uniform convergence, whether with respect to one variable or to two variables, is that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ for every point of the region, boundaries included. This fact is of vital importance. If, namely, we have a function defined as before, and if $\lim\limits_{\substack{y=y_0\\ z=z_0}} f(x, y, z)=f(x, y_0, z_0)$, and if we know that the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$ only for all points of the interval $\alpha<x<\beta$, where the endpoints are not included, then we do not know that the simultaneous limit also exists at the endpoints, and hence we cannot say that the function $f(x, y, z)$ converges uniformly throughout the interval endpoints included. All we can in this case say is that $f(x, y, z)$ converges uniformly throughout the interval $\alpha+\delta\leqq x\leqq\beta-\delta$, where δ is some positive number different from zero. The same considerations hold for uniform convergence with respect to two variables. The following example illustrates this.

Example 1.

Let $f(x, y, z)=\dfrac{xyz}{x^3+y^3+z^3}$, where $f(0, 0, 0)=0$, for the region

$$0\leqq x\leqq 1,\ 0\leqq y\leqq 1,\ 0<z\leqq 1.$$

This function is continuous in two variables taken together and in the third variable alone as has already been shown. Moreover the simultaneous limit $\lim\limits_{\substack{x=0\\ y=0\\ z=0}} f(x, y, z)$ exists and equals $f(0, 0, 0)$ for every (x, y) of the region $0<x\leqq 1$, $0<y\leqq 1$, but it does not exist for every (x, y) of the region when we include the boundaries. Hence the function does not converge uniformly throughout the region, boundaries included. In order to see what this means

geometrically, let us plot a few of the approximation surfaces. We have the following table of corresponding values:

x	y	z	$f(x, y, z)$
0	0	1	0
1	1	1	1/3
1	1/2	1	4/17
1/2	1/2	1	1/5
1	1	1/2	4/17
1	1/2	1/2	1/5
1/2	1/2	1/2	1/3
1	1	1/3	9/55
1	1/2	1/3	36/251
1	1/3	1/3	3/29
1/2	1/2	1/3	9/31
1/3	1/3	1/3	1/3

This gives us the following approximation surfaces for $z = 1$, $z = 1/2$, $z = 1/3$, see figure 9.

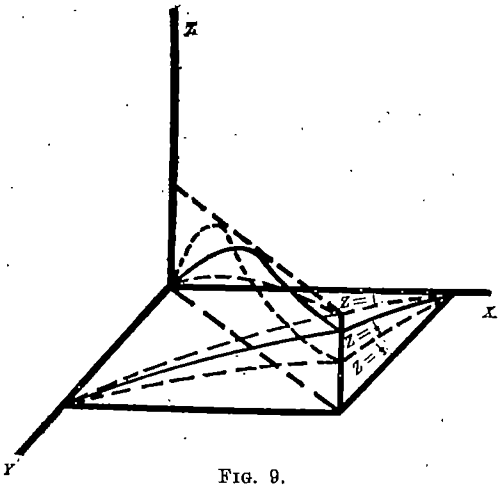

Fig. 9.

We see from this that each approximation surface has a peak of altitude 1/3 which moves along the line $y = x$ toward the origin as z approaches zero. At any point of the region, excepting the origin, we can therefore make $f(x, y, z)$ approach as closely as we please to zero by a suitable choice of z, but at the origin we cannot do so, and hence the function does not satisfy the condition of uniform convergence.

Having thus defined the two kinds of uniform convergence for a function of three variables, the question arises as to whether one of these will necessitate the other. That is, if $f(x, y, z)$ converges uniformly with respect to one variable, will it then also converge uniformly with respect to two variables, and conversely? This must be answered in the affirmative; for if $f(x, y, z)$ converges uniformly with respect to z, then we have

$$\lim_{z=z_0} f(x, y, z) = f(x, y, z_0) \tag{1}$$

for every (x, y) of the region, boundaries included. We also have

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = f(x_0, y_0, z_0) \tag{2}$$

at every point (x_0, y_0, z_0) of the region, boundaries included. From this last, however, follows that

$$\lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = f(x_0, y_0, z) \tag{3}$$

for every z of its interval. Now (1), (2), and (3) are exactly the conditions that $f(x, y, z)$ shall converge uniformly with respect to the two variables x and y. Hence it follows that if $f(x, y, z)$ converges uniformly with respect to one variable, then it will also converge uniformly with respect to two variables. The converse is proved in a similar manner.

We have given two different definitions of uniform convergence and have asserted that they are equivalent to each other. We shall now prove that this is in fact the case.

Theorem 1.—If the function $f(x, y, z)$ is continuous in two variables taken together and in the third variable alone, and if we have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) of $\alpha \leq x \leq \beta$ and $a \leq y \leq b$, then the necessary and sufficient condition that $f(x, y, z)$ converges uniformly to $f(x, y, z_0)$ throughout the region $\alpha \leq x \leq \beta$, $a \leq y \leq b$, is that at every point of this region, boundaries included, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ shall exist and equal $f(x_0, y_0, z_0)$.

The condition is necessary; for let (x_0, y_0) be any point of the region and choose an arbitrarily small $\sigma > 0$. Now since $f(x, y, z)$ converges uniformly throughout the region, there exists a $\theta_0 > 0$, such that

$$|f(x, y, z_k) - f(x, y, z_0)| < 1/4\sigma \tag{1}$$

for every z_k between z_0 and $z_0 + \theta_0$ and for every point (x, y) of the region boundaries included. Furthermore, since $f(x, y, z)$ is continuous in (x, y) together, we can in the plane $z = z_k$ construct a circle of radius r_0 about (x_0, y_0), such that for every point (x, y, z_k) of this circle we have

$$|f(x_0, y_0, z_k) - f(x, y, z_k)| < 1/4\sigma. \tag{2}$$

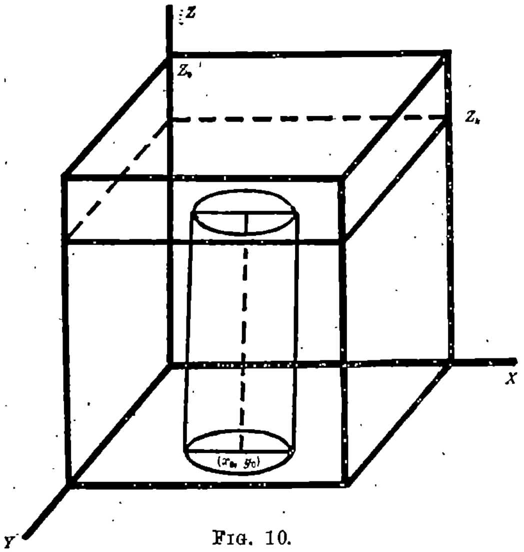

FIG. 10.

Now construct a right cylinder having this circle as base and its elements parallel to the z-axis, the bases of the cylinder lying in the planes $z = z_0$ and $z = z_k$, see figure 10.

Let (x, y, z) be any point within this cylinder. Since $f(x, y, z)$ is continuous in z alone we have by taking z_k near enough to z_0

$$|f(x, y, z_0) - f(x, y, z)| < 1/4\sigma \tag{3}$$

and

$$|f(x_0, y_0, z_0) - f(x_0, y_0, z_k)| < 1/4\sigma. \tag{4}$$

Adding these four inequalities we have

$$|f(x_0, y_0, z_0) - f(x, y, z)| < \sigma,$$

where (x, y, z) is any point within the cylinder. Hence the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists at (x_0, y_0, z_0) and equals $f(x_0, y_0, z_0)$. But (x_0, y_0) is any point of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, and hence the simultaneous limit exists at every point of the region.

The condition is also sufficient; for since at any point (x_0, y_0, z_0) of our region, boundaries included, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists and equals $f(x_0, y_0, z_0)$, we can first choose an arbitrarily small $\sigma > 0$ and can then construct a sphere R_0 of radius r_0 about (x_0, y_0, z_0) as center, such that there shall exist no point (ξ, η, ζ) within this sphere for which

$$|f(x_0, y_0, z_0) - f(\xi, \eta, \zeta)| \leqq \sigma. \tag{5}$$

For different points (x, y) of the region, but for the same value of σ, the radii of the resulting spheres may vary and are functions of (x, y). As such they have a lower limit ρ' different from zero, for the entire region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$; for suppose that these radii approach zero as limit as (x, y) approaches some point (x_0, y_0) of the region $(\alpha, \beta; a, b)$. Then it would be impossible to find any region about (x_0, y_0, z_0) which is free from points (ξ, η, ζ) that satisfy (5). Furthermore, the limiting point (x_0, y_0, z_0) must also be a point of the region. The radii ρ therefore have a definite lower limit ρ' on on the region $(\alpha, \beta; a, b)$ and this may therefore serve for the whole region, so that the definition for uniform convergence is satisfied.

9. *Continuity of the Limiting Function.*—Thus far we have not said anything about the continuity of the limiting functions $f(x, y, z_0)$ and $f(x, y_0, z_0)$. We now take up theorems that deal with this.

Theorem 1.—If the function $f(x, y, z)$ converges uniformly with respect to z, that is, if it converges uniformly to $f(x, y, z_0)$ throughout the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, then $f(x, y, z_0)$ is continuous in (x, y) together throughout this region.

Let (x_0, y_0) be any point of the given region. Since $f(x, y, z)$ converges uniformly throughout the region, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ exists for every point of the region, boundaries included, and equals $f(x_0, y_0, z_0)$. But we also have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) in the neighborhood of (x_0, y_0). Hence we have by theorem 4 of section 2

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = \lim_{\substack{x=x_0\\ y=y_0}} \lim_{z=z_0} f(x, y, z) = \lim_{z=z_0} \lim_{\substack{x=x_0\\ y=y_0}} f(x, y, z) = f(x_0, y_0, z_0).$$

From this follows at once that $f(x, y, z_0)$ is continuous in (x, y) together at (x_0, y_0). But since (x_0, y_0) is any point of the region, boundaries included, it follows that $f(x, y, z_0)$ is continuous in (x, y) throughout the entire region.

Theorem 2. If the function $f(x, y, z)$ converges uniformly with respect to y and z, that is if it converges uniformly to $f(x, y_0, z_0)$ in the interval $\alpha \leqq x \leqq \beta$, then $f(x, y_0, z_0)$ is continuous in x throughout the interval including the boundaries.

The proof is similar to that of theorem 1. Let x_0 be any point of the interval $\alpha \leqq x \leqq \beta$. Since $f(x, y, z)$ converges uniformly to $f(x, y_0, z_0)$ throughout the interval, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z)$ must exist at every point of the interval, end points included, and must equal $f(x_0, y_0, z_0)$. But we have also $\lim\limits_{\substack{y=y_0\\ z=z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval. Hence we have by theorem 3 of section 2

$$\lim_{\substack{x=x_0\\ y=y_0\\ z=z_0}} f(x, y, z) = \lim_{x=x_0} \lim_{\substack{y=y_0\\ z=z_0}} f(x, y, z) = \lim_{\substack{y=y_0\\ z=z_0}} \lim_{x=x_0} f(x, y, z) = f(x_0, y_0, z_0).$$

From this follows at once that $f(x, y_0, z_0)$ is continuous in x at the point $x = x_0$. But x_0 is any point of the interval $\alpha \leqq x \leqq \beta$, and hence $f(x, y_0, z_0)$ must be continuous in x throughout the interval, end points included.

Remark.—Theorems 1 and 2 give us sufficient but not necessary conditions for the continuity of the limiting functions $f(x, y, z_0)$ and $f(x, y_0, z_0)$. We can show this by means of an example which has already been used several times.

Let $f(x, y, z) = \dfrac{xyz}{x^3 + y^3 + z^3}$, where $f(0, 0, 0) = 0$. Here $f(x, y, 0) = 0$

throughout the region $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and it is therefore continuous in (x, y) together at the point $(0, 0)$. Likewise $f(x, 0, 0) = 0$ throughout the interval $0 \leqq x \leqq 1$, and it is therefore continuous in x at the point $x = 0$. But the function $f(x, y, z)$ does not converge uniformly either to $f(x, y, 0)$ or to $f(x, 0, 0)$ as has already been shown.

It is, however, possible to obtain a necessary and sufficient condition that the limiting function $f(x, y, z_0)$ is continuous in (x, y) together at any point (x, y) of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, and that the limiting function $f(x, y_0, z_0)$ is continuous in x at any point x of the interval $\alpha \leqq x \leqq \beta$.

Theorem 3.[*]—If the function $f(x, y, z)$ is finite and continuous in (x, y) together and in z alone throughout the region

$$\alpha \leqq x \leqq \beta,\ a \leqq y \leqq b,\ z_0 < z \leqq z_n,$$

then the necessary and sufficient condition that the limiting function $f(x, y, z_0)$ is finite and also continuous in (x, y) together at any fixed point (x_0, y_0) of the region

$$\alpha \leqq x \leqq \beta,\ a \leqq y \leqq b, \qquad \text{(A)}$$

is

(1) that $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every constant pair of values (x, y) of region (A).

(2) that having chosen an arbitrarily small $\sigma > 0$, there shall then exist a set of z's $(z = z_1, z_2, z_3, \ldots)$ dense at $z = z_0$, and that corresponding to each z_k there shall exist a circle of radius r_k about (x_0, y_0), where however r_k may vary with z_k, such that for all points (x, y) within this circle

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma.$$

We first prove that the condition is necessary. Our hypothesis therefore is that $f(x, y, z_0)$ is finite and continuous in (x, y) together at (x_0, y_0) where (x_0, y_0) is some fixed point of region (A). Hence we can choose an arbitrarily small $\sigma > 0$ and can then, in the plane $z = z_0$, find a circle of radius r_0 about (x_0, y_0), such that for every point (x, y) of this circle we have

$$|f(x, y, z_0) - f(x_0, y_0, z_0)| < 1/3\,\sigma \qquad (1)$$

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[*] Arzela proves this theorem for the case of a function of two variables. See "Sulle serie di funzioni." Accademia delle sc. di Bologna, 1899, page 142.

By hypothesis we also have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$, and hence also $\lim\limits_{z_k=z_0} f(x, y, z_k) = f(x, y, z_0)$, for every (x, y) of region (A). From this follows that for the same σ as above we can determine a $\delta > 0$, such that

$$|f(x_0, y_0, z_k) - f(x_0, y_0, z_0)| < 1/3\,\sigma \tag{2}$$

where $$z_0 < z \leqq z_0 + \delta.$$

We also have that $f(x, y, z)$ is continuous in (x, y) together, when z is constant, at every point of region (A), and hence we can in the plane $z = z_k$ determine a circle of radius r_k about (x_0, y_0), such that for every point of this circle

$$|f(x, y, z_k) - f(x_0, y_0, z_k)| < 1/3\,\sigma \tag{3}$$

where $$z_0 < z \leqq z_0 + \delta.$$

Inequalities (1), (2), and (3) hold for all values of z and z_k between z_0 and $z_0 + \delta$. If $r_k < r_0$, then (3) holds for all points (x, y) of a circle of radius r_k about (x_0, y_0) in the plane $z = z_k$. We may then add these inequalities and obtain

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma,$$

and hence the condition is necessary.

We next prove that the condition is also sufficient. By hypothesis we now have that $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ for every point (x, y) of region (A). Hence we can choose at pleasure $\sigma > 0$ and can then determine $\delta > 0$, such that

$$|f(x_0, y_0, z_s) - f(x_0, y_0, z_0)| < 1/4\,\sigma \tag{4}$$

where $$z_0 < z_s \leqq z_0 + \delta.$$

But $f(x, y, z)$ is continuous in (x, y) together at every point of region (A), and hence we can for the same σ as above find, in the plane $z = z_s$, a circle of radius r_s about (x_0, y_0), such that for all points of this circle

$$|f(x, y, z_s) - f(x_0, y_0, z_s)| < 1/4\,\sigma \tag{5}$$

where $$z_0 < z_s \leqq z_0 + \delta.$$ (See figure 11.)

From the second part of the hypothesis follows that in the plane $z = z_k$ there exists a circle of radius $\bar{r}_k$ about (x_0, y_0), such that for all points of this circle

$$|f(x, y, z_0) - f(x, y, z_k)| < 1/4\,\sigma \tag{6}$$

where $$z_0 < z_k \leqq z_0 + \delta.$$

It is to be noticed that if $z_s = z_k$, then it is not necessary that also $r_s = \bar{r}_k$, for the two circles do not arise from the same conditions. We may however choose r_s to fit both cases if $r_s < \bar{r}_k$. Since $\lim_{z_k = z_0} f(x, y, z_k) = f(x, y, z_0)$ for every constant pair of values (x, y) of region (A), we have for the same σ as above

$$|f(x_0, y_0, z_k) - f(x_0, y_0, z_0)| < 1/4\,\sigma \tag{7}$$

where

$$z_0 < z_k \leqq z_0 + \delta.$$

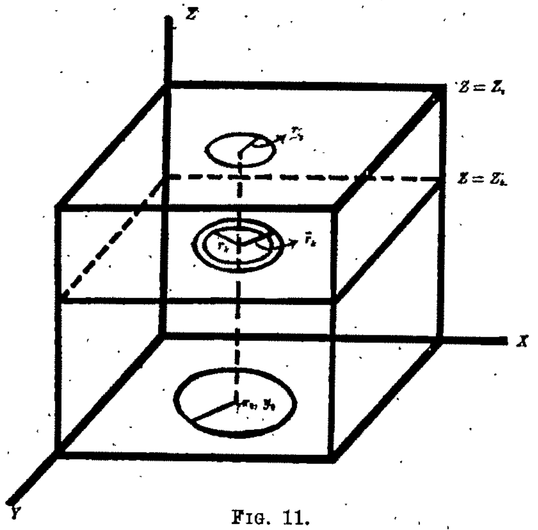

FIG. 11.

The continuity of $f(x, y, z)$ in (x, y) together throughout region (A) also gives us

$$|f(x, y, z_k) - f(x_0, y_0, z_k)| < 1/4\,\sigma \tag{8}$$

where

$$z_0 < z_k \leqq z_0 + \delta,$$

for every point (x, y) of the circle of radius $\bar{r}_k$ about (x_0, y_0) in the plane $z = z_k$. We will suppose that $r_s < \bar{r}_k$, so that inequality (8) holds for all points of the circle of radius r_s. Inequalities 4, 5, 7, and 8, now hold for every z where $z_0 < z_k \leqq z_0 + \delta$, and for all points (x, y) of the circle of radius r_s. We may therefore add them and obtain

$$|f(x, y, z_s) - f(x, y, z_k)| < \sigma \tag{9}$$

where

$$z_0 < z_s \leqq z_0 + \delta,$$

and for all points (x, y) of a circle of radius r_s about (x_0, y_0). We can now choose

By hypothesis we also have $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$, and hence also $\lim\limits_{z_k = z_0} f(x, y, z_k) = f(x, y, z_0)$, for every (x, y) of region (A). From this follows that for the same σ as above we can determine a $\delta > 0$, such that

$$|f(x_0, y_0, z_k) - f(x_0, y_0, z_0)| < 1/3\,\sigma \tag{2}$$

where $$z_0 < z \leqq z_0 + \delta.$$

We also have that $f(x, y, z)$ is continuous in (x, y) together, when z is constant, at every point of region (A), and hence we can in the plane $z = z_k$ determine a circle of radius r_k about (x_0, y_0), such that for every point of this circle

$$|f(x, y, z_k) - f(x_0, y_0, z_k)| < 1/3\,\sigma \tag{3}$$

where $$z_0 < z \leqq z_0 + \delta.$$

Inequalities (1), (2), and (3) hold for all values of z and z_k between z_0 and $z_0 + \delta$. If $r_k < r_0$, then (3) holds for all points (x, y) of a circle of radius r_k about (x_0, y_0) in the plane $z = z_k$. We may then add these inequalities and obtain

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma,$$

and hence the condition is necessary.

We next prove that the condition is also sufficient. By hypothesis we now have that $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every point (x, y) of region (A). Hence we can choose at pleasure $\sigma > 0$ and can then determine $\delta > 0$, such that

$$|f(x_0, y_0, z_s) - f(x_0, y_0, z_0)| < 1/4\,\sigma \tag{4}$$

where $$z_0 < z_s \leqq z_0 + \delta.$$

But $f(x, y, z)$ is continuous in (x, y) together at every point of region (A), and hence we can for the same σ as above find, in the plane $z = z_s$, a circle of radius r_s about (x_0, y_0), such that for all points of this circle

$$|f(x, y, z_s) - f(x_0, y_0, z_s)| < 1/4\,\sigma \tag{5}$$

where $$z_0 < z_s \leqq z_0 + \delta.$$ (See figure 11.)

From the second part of the hypothesis follows that in the plane $z = z_k$ there exists a circle of radius $\bar{r}_k$ about (x_0, y_0), such that for all points of this circle

$$|f(x, y, z_0) - f(x, y, z_k)| < 1/4\,\sigma \tag{6}$$

where $$z_0 < z_k \leqq z_0 + \delta.$$

It is to be noticed that if $z_s = z_k$, then it is not necessary that also $r_s = \bar{r}_k$, for the two circles do not arise from the same conditions. We may however choose r_s to fit both cases if $r_s < \bar{r}_k$. Since $\lim\limits_{z_k = z_0} f(x, y, z_k) = f(x, y, z_0)$ for every constant pair of values (x, y) of region (A), we have for the same σ as above

$$|f(x_0, y_0, z_k) - f(x_0, y_0, z_0)| < 1/4\,\sigma \tag{7}$$

where $$z_0 < z_k \leqq z_0 + \delta.$$

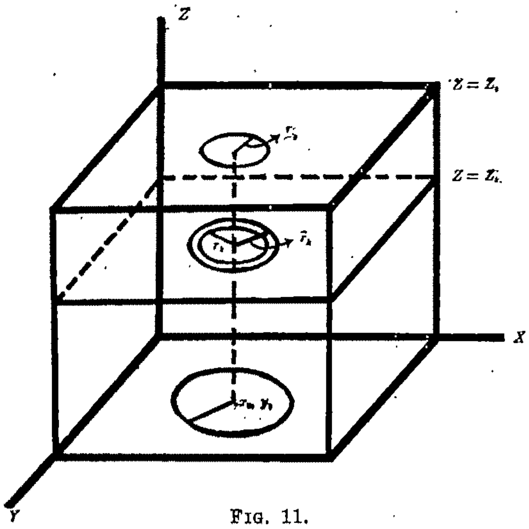

FIG. 11.

The continuity of $f(x, y, z)$ in (x, y) together throughout region (A) also gives us

$$|f(x, y, z_k) - f(x_0, y_0, z_k)| < 1/4\,\sigma \tag{8}$$

where $$z_0 < z_k \leqq z_0 + \delta,$$

for every point (x, y) of the circle of radius $\bar{r}_k$ about (x_0, y_0) in the plane $z = z_k$. We will suppose that $r_s < \bar{r}_k$, so that inequality (8) holds for all points of the circle of radius r_s. Inequalities 4, 5, 7, and 8, now hold for every z where $z_0 < z_k \leqq z_0 + \delta$, and for all points (x, y) of the circle of radius r_s. We may therefore add them and obtain

$$|f(x, y, z_s) - f(x, y, z_k)| < \sigma \tag{9}$$

where $$z_0 < z_s \leqq z_0 + \delta,$$

and for all points (x, y) of a circle of radius r_s about (x_0, y_0). We can now choose

δ in such a manner that 4, 5, and 9, shall hold for every z where $z_0 < z_s \leqq z_0 + \delta$, so that we may add these inequalities and thus obtain

$$|f(x, y, z_0) - f(x_0, y_0, z_0)| < 3/2\,\sigma \qquad (10)$$

where (x, y) is any point within the circle of radius r_s. Hence there exists in the plane $z = z_0$ about (x_0, y_0) a circle of radius greater than zero, such that (10) holds at every point within this circle. Hence $f(x, y, z_0)$ is continuous in (x, y) together at (x_0, y_0). This proves the theorem.

We can obtain an analogous theorem for the continuity of the limiting function $f(x, y_0, z_0)$ at the point $x = x_0$.

Theorem 4. If the function $f(x, y, z)$ is finite and continuous in x alone and in (y, z) together at every point of the region

$$\alpha \leqq x \leqq \beta, \; y_0 < y \leqq y_m, \; z_0 < z \leqq z_n,$$

then the necessary and sufficient condition that the limiting function $f(x, y_0, z_0)$ is finite and continuous in x at x_0 where x_0 is some fixed point of the interval

$$\alpha \leqq x \leqq \beta, \qquad (A)$$

is

(1) that $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval (A);

(2) that having chosen an arbitrarily small $\sigma > 0$, there shall then exist a set of pairs of values (y_i, z_k) which shall be dense at (y_0, z_0) and corresponding to each pair there shall exist an interval greater than zero, such that for every x of this interval

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

The proof of this theorem is similar to that of the preceding theorem. Since $f(x, y_0, z_0)$ is continuous in x at the point $x = x_0$, we can choose an arbitrarily small $\sigma > 0$ and can then determine an interval $(x_0 - \delta_1, x_0 + \delta_2)$ greater than zero, such that

$$|f(x_0, y_0, z_0) - f(x, y_0, z_0)| < 1/3\,\sigma \qquad (1)$$

for every x of this interval. Since $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ and hence also

$\lim\limits_{\substack{y_i = y_0 \\ z_k = z_0}} f(x, y_i, z_k) = f(x, y_0, z_0)$ for every x of the interval (A), we can for the same σ find some value (y_s, z_t), such that

$$|f(x_0, y_i, z_k) - f(x_0, y_0, z_0)| < 1/3\,\sigma \tag{2}$$

where $y_0 < y_i \leqq y_s, \ z_0 < z_k \leqq z_t$.

But $f(x, y, z)$ is continuous in x when y and z are constant, hence for the same σ we can determine a $\bar{\delta}_1$ and a $\bar{\delta}_2$ each greater than zero, such that

$$|f(x, y_i, z_k) - f(x_0, y_i, z_k)| < 1/3\,\sigma \tag{3}$$

where $x_0 - \bar{\delta}_1 < x < x_0 + \bar{\delta}_2$.

Now let δ_1 and δ_2 be less than $\bar{\delta}_1$ and $\bar{\delta}_2$ respectively. Then we may add (1), (2), and (3), and thus obtain

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma$$

where $x_0 - \delta_1 < x < x_0 + \delta_2$.

This proves that the condition is necessary.

The condition is also sufficient. We now have by hypothesis that $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval (A). Hence we can choose at pleasure a $\sigma > 0$ and can then find a δ_1 and a δ_2 each greater than zero, such that

$$|f(x_0, y_0, z_0) - f(x_0, y_s, z_t)| < 1/6\,\sigma \tag{4}$$

where $y_0 < y_s < y_0 + \delta_1$ and $z_0 < z_t < z_0 + \delta_2$.

Since $f(x, y, z)$ is continuous in x, we can for the same σ and on the line $(x = y_s, \ z = z_t)$ find about x an interval $(x_0 - \delta_s, \ x_0 + \delta_s)$, such that

$$|f(x_0, y_s, z_t) - f(x, y_s, z_t)| < 1/6\,\sigma \tag{5}$$

where $x_0 - \delta_s < x < x_0 + \delta_s$.

From the second part of the hypothesis it follows that on the line $(y = y_i, \ z = z_k)$ we can for the same value of σ determine an interval $(x_0 - \delta_k, \ x_0 + \delta_k)$, such that

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < 1/6\,\sigma \tag{6}$$

where $x_0 - \delta_k < x < x_0 + \delta_k, \ \ y_0 < y_i \leqq y_0 + \delta_1, \ \ z_0 < z_k \leqq z_0 + \delta_2$.

If $y_i = y_s$ and $z_k = z_t$, then it is not necessary that also $\delta_s = \delta_k$. But if $\delta_s < \delta_k$, then δ_s will fit both cases. Let us choose it thus. Since $\lim\limits_{\substack{y_i = y_0 \\ z_k = z_0}} f(x, y_i, z_k) = f(x, y_0, z_0)$ for every x of the interval (A), we can for the same value of σ determine a $\bar{\delta}_1$ and a $\bar{\delta}_2$, such that

$$|f(x_0, y_i, z_k) - f(x_0, y_0, z_0)| < 1/6\,\sigma \tag{7}$$

where $\qquad y_0 < y_i \leqq y_0 + \bar{\delta}_1, \quad z_0 < z_k \leqq z_0 + \bar{\delta}_2.$

Finally we have from the continuity of $f(x, y, z)$ in x that for the same σ we can find a $\bar{\delta}_k > 0$, such that

$$|f(x, y_i, z_k) - f(x_0, y_i, z_k)| < 1/6\,\sigma \tag{8}$$

where $\qquad x_0 - \bar{\delta}_k < x < x_0 + \bar{\delta}_k.$

Now let δ_s be less than δ_k, and let δ_1 and δ_2 be less than $\bar{\delta}_1$ and $\bar{\delta}_2$ respectively. Then (4), (5), (7), and (8), hold for the same intervals and we may add them. We thus obtain

$$|f(x, y_s, z_t) - f(x, y_i, z_k)| < 2/3\,\sigma \tag{9}$$

where $y_0 < y \leqq y_0 + \delta_1$ and $z_0 < z \leqq z_0 + \delta_2$, and for all points x of the interval $(x_0 - \delta_s, x_0 + \delta_s)$. We can now choose δ_1 and δ_2 in such a manner that the inequalities (4), (5), and (9), hold simultaneously, so that we may add these and thus obtain

$$|f(x, y_0, z_0) - f(x_0, y_0, z_0)| < \sigma \tag{10}$$

where x is any point of the interval $(x_0 - \delta, x_0 + \delta)$. Hence on the line $(y = y_0, z = z_0)$ there exists about x_0 an interval greater than zero for every point of which inequality (10) holds. Hence $f(x, y_0, z_0)$ is continuous in x at the point $x = x_0$, which proves the theorem.

We have thus obtained necessary and sufficient conditions that the functions $f(x, y, z_0)$ and $f(x, y_0, z_0)$ are continuous in (x, y) and x respectively at some fixed point. We next wish to find the condition that they shall be continuous throughout a whole region or interval. Such a condition is that the conditions of theorems 3 and 4 hold for every point of the region or interval in question. We can, however, obtain another condition which will be of use later on.

Before giving this condition we must first define another kind of uniform convergence. This is Arzela's "*Convergenza uniforme a tratti and a stratti.*"*

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*See Arzela: Sulle serie di funzioni. Accademia delle scienze di Bologna, 1899, p. 158.

We may translate this by uniform convergence by segments and by areas. We first give the definition for a function of two variables.

The function $f(x, y)$ shall be defined for all points of the region $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_n$, and shall be continuous in each variable alone throughout this region; furthermore $\lim\limits_{y = y_0} f(x, y)$ shall equal $f(x, y_0)$ for every x for which it is defined. If now there exists between y_0 and y_n a finite number of lines $(y = y_1, y_2, y_3, \ldots, y_p)$ and on each of these a finite number of segments each greater than or equal to some finite length L which is itself greater than zero, such that the projections of these segments on $y = y_0$ completely fill the interval (α, β) and that for every point of these segments

$$| f(x, y_0) - f(x, y_k) | < \sigma,$$

then we say that $f(x, y)$ converges uniformly by segments to $f(x, y_0)$ throughout the interval.

In an analogous manner we define uniform convergence by areas for a function of three variables. Let the function $f(x, y, z)$ be defined for every point of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, $z_0 < z \leqq z_n$, and let it be continuous in (x, y), together and in z alone throughout the region; furthermore let $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every pair of values (x, y) of the region. If now there exists between z_0 and z_k a finite number of planes $(z = z_1, z_2, z_3, \ldots, z_p)$ and on each of these planes a finite number of rectangles having their sides parallel to the x and y axes and such that their dimensions are each greater than or equal to some quantity greater than zero, and furthermore such that the projections of these rectangles on the plane $z = z_0$ completely fill the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, and that for all points (x, y) of these rectangles the following inequality holds

$$| f(x, y, z_0) - f(x, y, z_k) | < \sigma,$$

then we say that $f(x, y, z)$ converges uniformly by areas to $f(x, y, z_0)$ throughout the region.

We may also define such uniform convergence for the case that $f(x, y, z)$ converges to $f(x, y_0, z_0)$. Let $f(x, y, z)$ be defined for every point of the region $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_m$, $z_0 < z \leqq z_n$, and let it be continuous throughout this region in (y, z) together and in x alone. Furthermore let $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$

If $y_i = y_s$ and $z_k = z_t$, then it is not necessary that also $\delta_s = \delta_k$. But if $\delta_s < \delta_k$, then δ_s will fit both cases. Let us choose it thus. Since

$$\lim_{\substack{y_i = y_0 \\ z_k = z_0}} f(x, y_i, z_k) = f(x, y_0, z_0)$$

for every x of the interval (A), we can for the same value of σ determine a $\bar{\delta}_1$ and a $\bar{\delta}_2$, such that

$$|f(x_0, y_i, z_k) - f(x_0, y_0, z_0)| < 1/6\,\sigma \tag{7}$$

where $\qquad y_0 < y_i \leqq y_0 + \bar{\delta}_1, \quad z_0 < z_k \leqq z_0 + \bar{\delta}_2.$

Finally we have from the continuity of $f(x, y, z)$ in x that for the same σ we can find a $\bar{\delta}_k > 0$, such that

$$|f(x, y_i, z_k) - f(x_0, y_i, z_k)| < 1/6\,\sigma \tag{8}$$

where $\qquad x_0 - \bar{\delta}_k < x < x_0 + \bar{\delta}_k.$

Now let δ_s be less than δ_k, and let δ_1 and δ_2 be less than $\bar{\delta}_1$ and $\bar{\delta}_2$ respectively. Then (4), (5), (7), and (8), hold for the same intervals and we may add them. We thus obtain

$$|f(x, y_s, z_l) - f(x, y_i, z_k)| < 2/3\,\sigma \tag{9}$$

where $y_0 < y \leqq y_0 + \delta_1$ and $z_0 < z \leqq z_0 + \delta_2$, and for all points x of the interval $(x_0 - \delta_s, x_0 + \delta_s)$. We can now choose δ_1 and δ_2 in such a manner that the inequalities (4), (5), and (9), hold simultaneously, so that we may add these and thus obtain

$$|f(x, y_0, z_0) - f(x_0, y_0, z_0)| < \sigma \tag{10}$$

where x is any point of the interval $(x_0 - \delta, x_0 + \delta)$. Hence on the line $(y = y_0, z = z_0)$ there exists about x_0 an interval greater than zero for every point of which inequality (10) holds. Hence $f(x, y_0, z_0)$ is continuous in x at the point $x = x_0$, which proves the theorem.

We have thus obtained necessary and sufficient conditions that the functions $f(x, y, z_0)$ and $f(x, y_0, z_0)$ are continuous in (x, y) and x respectively at some fixed point. We next wish to find the condition that they shall be continuous throughout a whole region or interval. Such a condition is that the conditions of theorems 3 and 4 hold for every point of the region or interval in question. We can, however, obtain another condition which will be of use later on.

Before giving this condition we must first define another kind of uniform convergence. This is Arzela's "*Convergenza uniforme a tratti and a stratti.*" *

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* See Arzela: Sulle serie di funzioni. Accademia delle scienze di Bologna, 1899, p. 153.

We may translate this by uniform convergence by segments and by areas. We first give the definition for a function of two variables.

The function $f(x, y)$ shall be defined for all points of the region $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_n$, and shall be continuous in each variable alone throughout this region; furthermore $\lim\limits_{y = y_0} f(x, y)$ shall equal $f(x, y_0)$ for every x for which it is defined. If now there exists between y_0 and y_n a finite number of lines ($y = y_1, y_2, y_3, \ldots, y_p$) and on each of these a finite number of segments each greater than or equal to some finite length L which is itself greater than zero, such that the projections of these segments on $y = y_0$ completely fill the interval (α, β) and that for every point of these segments

$$| f(x, y_0) - f(x, y_k) | < \sigma,$$

then we say that $f(x, y)$ converges uniformly by segments to $f(x, y_0)$ throughout the interval.

In an analogous manner we define uniform convergence by areas for a function of three variables. Let the function $f(x, y, z)$ be defined for every point of the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, $z_0 < z \leqq z_n$, and let it be continuous in (x, y), together and in z alone throughout the region; furthermore let $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every pair of values (x, y) of the region. If now there exists between z_0 and z_k a finite number of planes ($z = z_1, z_2, z_3, \ldots, z_p$) and on each of these planes a finite number of rectangles having their sides parallel to the x and y axes and such that their dimensions are each greater than or equal to some quantity greater than zero, and furthermore such that the projections of these rectangles on the plane $z = z_0$ completely fill the region $\alpha \leqq x \leqq \beta$, $a \leqq y \leqq b$, and that for all points (x, y) of these rectangles the following inequality holds

$$| f(x, y, z_0) - f(x, y, z_k) | < \sigma,$$

then we say that $f(x, y, z)$ converges uniformly by areas to $f(x, y, z_0)$ throughout the region.

We may also define such uniform convergence for the case that $f(x, y, z)$ converges to $f(x, y_0, z_0)$. Let $f(x, y, z)$ be defined for every point of the region $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_m$, $z_0 < z \leqq z_n$, and let it be continuous throughout this region in (y, z) together and in x alone. Furthermore let $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$

for every constant value x of the interval. If now we have a finite set of values (y_i, z_k), where the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ for each (y_i, z_k) is less than the corresponding value for the preceding (y_i, z_k), and if on each line $(y = y_i, z = z_k)$ there exists a finite number of segments, each of length greater than or equal to L, where L is itself greater than zero, such that the projections of these segments on the line $(y = y_0, z = z_0)$ completely fill the interval (α, β) and that for every point x of these segments we have

$$|f(x, y_0, z_0) - f(x, y_i, z_k)| < \sigma,$$

then we say that $f(x, y, z)$ converges uniformly by segments to $f(x, y_0, z_0)$ throughout the interval.

This new kind of uniform convergence is entirely distinct from both the ordinary uniform convergence and from Dini's *"einfach gleichmässige Convergenz"*, as Arzela[*] has shown in the case of a function of two variables. For a function of three variables the distinction between these three kinds of uniform convergence is as follows.

In all three cases the function $f(x, y, z)$ is defined for all points of the region

$$\alpha \leqq x \leqq \beta, \; a \leqq y \leqq b, \; z_0 < z \leqq z_n,$$

and $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every point of the region

$$\alpha \leqq x \leqq \beta, \; a \leqq y \leqq b. \tag{A}$$

We then choose at pleasure a $\sigma > 0$.

In the case of ordinary uniform convergence we must then be able to find a $z_s \neq z_0$, such that for *every* z between z_s and z_0 and for every point (x, y) of region (A) we have

$$|f(x, y, z) - f(x, y, z_0)| < \sigma. \tag{1}$$

In the case of Dini's "einfach gleichmässigs Convergenz" there exists between z_s and z_0 only a set of z's dense at z_0, such that (1) holds for each of these z's and for every (x, y) of region (A).

In the case of Arzela's uniform convergence by areas there exists between z_s and z_0 only a finite number of planes $z = z_k$, and on each such plane a finite number of rectangles, each of area greater than or equal to G where G is finite and greater than zero, whose projections on the plane $z = z_0$ completely fill region (A), such that (1) holds only for all points of these rectangles.

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[*] Arzela: loco citato, page 153.

We are now ready to take up the theorems that deal with the continuity of the limiting function throughout the whole region.

Theorem 5.[*] If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$a \leqq x \leqq \beta,\ a \leqq y \leqq b,\ z_0 < z \leqq z_n,$$

and if $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every (x, y) of the region

$$a \leqq x \leqq \beta,\ a \leqq y \leqq b, \tag{A}$$

then the necessary and sufficient condition that $f(x, y, z_0)$ is continuous in (x, y) together throughout region (A) is that $f(x, y, z)$ shall converge uniformly by areas to $f(x, y, z_0)$ throughout (A).

The condition is necessary. Then our hypothesis is that $f(x, y, z_0)$ is continuous in (x, y) together at every point of region (A). If therefore (x', y') is any point of region (A), then it follows from theorem 3 of this section that there exists a set of z's $(z = z_1, z_2, z_3, \ldots)$ dense at $z = z_0$, and on each z_k sufficiently near to z_0 there exists about (x', y') a rectangle having its sides parallel to the axes and having an area δ_k different from zero, such that for every point (x, y) within this rectangle

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma, \tag{3}$$

where σ is an arbitrarily small positive number. It will be advantageous in the following not to consider this whole rectangle about (x', y'), but to divide it into four rectangles by lines parallel to the axes and to consider each of these rectangles separately. Let us take, for example, the rectangle $P'ABC$ to the right and front of (x', y'). See fig. 12.

We may then say that, having chosen an arbitrarily small $\sigma > 0$ and any z_s, there will then exist for every z_k between z_s and z_0 a rectangle of area greater than zero to the right and front of (x', y'), such that (1) holds for every point (x, y) of this rectangle. The area of this rectangle may vary with z_k and may in fact be zero for some values of z_k, but it cannot be zero for every value of z_k, unless (x', y') is a point of the boundaries $x = \beta$ and $y = b$. The area of this rectangle is moreover a function of z_k and as such it has an upper limit which is certainly less than the area of the region (A). Let us denote this upper limit by $\Delta(x', y')$. If now (x', y') is considered as a variable, then $\Delta(x', y')$ is a function of (x', y'). Consider now the region $\alpha \leqq x \leqq \beta - \varepsilon'$, $a \leqq y \leqq b - \varepsilon''$, where ε' and ε'' are two positive numbers chosen at pleasure, excepting that they shall be less than the corresponding dimensions of region (A). We can now show that within this new region $\Delta(x', y')$ considered as a function of (x, y) has a lower limit different from zero. Let this lower limit be denoted by l. There exists some point (x, y) of the region $\alpha \leqq x \leqq \beta - \varepsilon'$, $a \leqq y \leqq b - \varepsilon''$, such that the lower limit of $\Delta(x', y')$ is l in every neighborhood of this point (here $\Delta(x', y')$ is always the area of the rectangle to the right and front of the point).

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[*] Arzela: loco citato, page 147.

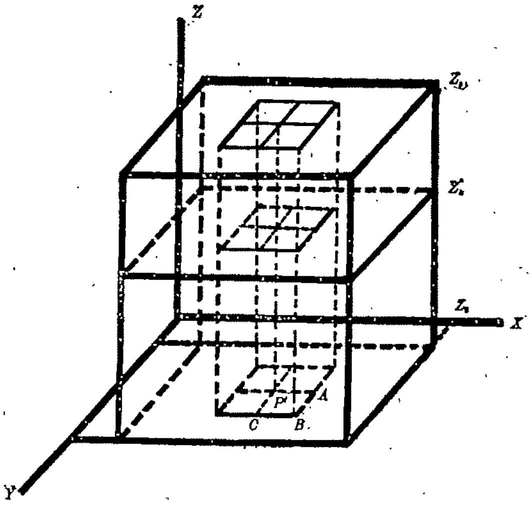

FIG. 13.

Let this point be (x_1, y_1). For this point the value of the function $\Delta(x_1, y_1)$ is certainly greater than zero. Now there surely exists among the values of z between z_8 and z_0 one value z_k for which the area of the rectangle, for every point (x, y) of which we have

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma,$$

is as near as we please to $\Delta(x_1, y_1)$. It is now evident that, if we consider on $z = z_0$ a rectangle of area $1/4\Delta(x_1, y_1)$, then we shall have for every point (x, y) of this rectangle at least a rectangle of area $1/4\Delta(x_1, y_1)$ to the right and front, such that for every point (x, y) of this rectangle the above inequality holds. Hence l must be greater than or equal to $1/4\Delta(x_1, y_1)$. But since this is greater than zero it follows that l is greater than zero.

In the same way we can show that the upper limit of each of the other three rectangles, into which we divided the original rectangle about (x', y'), when considered as a function of (x, y) has a lower limit greater than zero. Hence the entire rectangle about (x', y'), which is the upper limit for a fixed (x, y), has a lower limit greater than zero when considered as a function of (x, y).

We have thus shown that this lower limit exists and is greater than zero for the region (A) exclusive of the boundaries. It remains to take care of the boundaries. In order to do this we consider the region

$$\alpha + \varepsilon' \leqq x \leqq \beta - \varepsilon', \; a + \varepsilon'' \leqq y \leqq b - \varepsilon''. \qquad \text{(B)}$$

Let the lower limit of the area of the rectangle within the region (B) for every point of which

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma$$

be L. Now consider the remaining strip of region (A). It is evident that within this strip the dimensions of the rectangle for every point of which (1) holds cannot exceed ε' and ε''. Hence the lower limit L of the rectangle for the whole region (A) cannot exceed ε', ε''. Now let L' be some fixed positive quantity less than L. We can then say that, having chosen an arbitrarily small positive quantity σ and an arbitrary z_t, we can then find between z_t and z_0 a finite number of planes $z = z_k$, and on each a finite number of rectangles with their sides parallel to the axes, each rectangle having an area greater than or equal to L', such that for every point (x, y) of these rectangles

$$|f(x, y, z_k) - f(x, y, z_0)| < \sigma.$$

We thus prove that the condition of the theorem is necessary.

It remains to prove that it is also sufficient. Let us choose at pleasure a $\sigma > 0$ and any $z = z_t$. Let (x_0, y_0) be any point of region (A). Then it follows from the second part of the hypothesis that there exists between z_t and z_0 a finite number of planes $z = z_1, z_2, z_3, \ldots, z_p$, and on each a finite number of rectangles with sides parallel to the axes and each of area greater than zero, such that for all points (x, y) of these rectangles

$$|f(x, y, z) - f(x, y, z_0)| < \sigma. \qquad (1)$$

Since the projections of these rectangles on $z = z_0$ completely fill region (A) by hypothesis, it follows that the line through (x_0, y_0) perpendicular to the XY-plane must meet at least one of these rectangles. Let this be the rectangle on the plane $z = \bar{z}_1$. If now we let σ remain unchanged and choose $\bar{z}_1$ as a new z_t, then there exists between $\bar{z}_1$, and z_0 a set of planes finite in number and on each such plane a finite number of rectangles each of area greater than zero, such that for all points of these rectangles inequality (1) holds. Since again the projections of these rectangles on the plane $z = z_0$ completely fill region (A), the perpendicular through (x_0, y_0) to the XY-plane must meet at least one of these rectangles. Let it be a rectangle in the plane $z = \bar{z}_2$. Again let σ remain unchanged and choose $\bar{z}_2$ as our new z_t and repeat the reasoning. We thus obtain between z_t and z_0 a set of z's $(z = \bar{z}_1, \bar{z}_2, \bar{z}_3, \ldots)$ dense at $z = z_0$, and on each plane $z = \bar{z}_t$ a rectangle of area greater than zero about (x_0, y_0), such that for all points of these rectangles inequality (1) holds. Hence it follows from theorem 3 of this section that $f(x, y, z_0)$ is continuous in (x, y) together at the point (x_0, y_0). But (x_0, y_0) is any point of region (A) and therefore the function $f(x, y, z_0)$ is continuous in (x, y) together at every point of region (A). It is evident that if (x_0, y_0) had been a point of the boundary of region (A), then the rectangle in question could have been taken only up to the boundary.

We have an analogous theorem for the continuity of the limiting function $f(x, y_0, z_0)$. It is the following.

Theorem 6. If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha \leqq x \leqq \beta\,;\; y_0 < y \leqq y_m,\; z_0 < z \leqq z_n,$$

and if $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every x of the interval

$$\alpha \leqq x \leqq \beta, \tag{A}$$

then the necessary and sufficient condition that $f(x, y_0, z_0)$ is continuous in x throughout this interval is that $f(x, y, z)$ converges uniformly by segments to $f(x, y_0, z_0)$ throughout the interval.

The proof of this theorem is analogous to that of theorem 5. First the condition is necessary. Then we have by hypothesis that $f(x, y_0, z_0)$ is continuous in x throughout the interval (A) and we wish to prove that throughout this interval the function $f(x, y, z)$ converges uniformly by segments to $f(x, y_0, z_0)$. Since $f(x, y_0, z_0)$ is continuous in x throughout the interval (A), it follows from theorem 4 of this section that for any point x' of this interval there exists a set of values (y_i, z_k) dense at (y_0, z_0) and corresponding to each such pair of values there exists a $\delta_k > 0$, such that

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma \tag{1}$$

where $x' - \delta_k \leqq x \leqq x' + \delta_k$. It will be more accurate to consider separately the intervals to the left and to the right of x'. Having chosen an arbitrarily small positive number σ, we can then find a set of values (y_i, z_k) dense at (y_0, z_0) and corresponding to each pair (y_i, z_k) a positive δ_k, such that

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma$$

for every x of the interval of length δ_k to the right of x'. For every value (y_i, z_k) we obtain a corresponding δ_k. We choose in each case the largest possible value of the interval to the right of x', and it shall be this interval that is denoted by δ_k. It is then evident that δ_k cannot be zero for every pair of values (y_i, z_k) unless $x' = \beta$. Moreover δ_k is a function of (y_i, z_k) and as such it has an upper finite limit which is surely not greater than $\beta - \alpha$. Let us denote this upper limit by $\Delta(x')$. If now we let x' become variable, then $\Delta(x')$ is a function of x, which we shall consider in the interval $\alpha \leqq x \leqq \beta - \varepsilon$, where ε is any positive number less than $\beta - \alpha$. We can now assert that $\Delta(x')$ has a lower limit greater than zero when considered as a function of x in the interval $\alpha \leqq x \leqq \beta - \varepsilon$. Let l be the lower limit in question. Then there certainly exists some point x within the interval $(\alpha, \beta - \varepsilon)$, such that in every neighborhood of this point the lower limit of $\Delta(x')$ is l. Let x_1 be such a point. We then surely know that for $x = x_1$ the value $\Delta(x')$ of the interval to right must be greater than zero, since $x \neq \beta$. If therefore we consider the interval $x_1 \leqq x \leqq x_1 + 1/2\, \Delta(x_1)$, and if x'' is any point of this interval, then there certainly exists an interval of length at least equal to $1/2\, \Delta(x_1)$ to the right of x'', such that for every x of this interval.

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

Hence l must be at least equal to $1/2\, \Delta(x_1)$. This is however greater than zero and hence l is greater than zero. In a like manner we can show that in the interval $(\alpha + \varepsilon', \beta)$ there exists a similar lower limit l' greater than zero for the upper limits of $\Delta(x')$. It follows from this that for the interval $(\alpha + \varepsilon' \leqq x \leqq \beta - \varepsilon)$ there exists a lower limit L greater than zero for the total interval about x for which (1) holds.

From what has been said it follows that for the point $x = \beta$ there must exist a lower limit greater than zero of the intervals to the left. Call this Δ_β. Similarly there exists a lower limit greater than zero of the intervals to the right of the point $x = \alpha$. Call this Δ_α. Then there exists for the interval

$\alpha + \Delta_\alpha \leqq x \leqq \beta - \Delta_\beta$ a lower limit L greater than zero of the total interval about x for which (1) holds. For the whole interval $\alpha \leqq x \leqq \beta$ the lower limit of the corresponding interval is therefore not greater than $\Delta_\alpha + \Delta_\beta$. Denote this lower limit by L and let L' be some definite positive quantity less than L. It now follows from what has been said that if we choose an arbitrarily small positive σ and any pair of values (y_s, z_t) then there exists a finite number of lines $(y = y_i, z = z_k)$ where the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ is less than the absolute value of $\sqrt{(y_s^2 + z_t^2)}$, and on each such line there exists a finite number of segments each in length greater than L' and whose projections on $(y = y_0, z = z_0)$ completely fill the interval $\alpha \leqq x \leqq \beta$, such that for every point x of these segments

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

This proves that the condition of the theorem is necessary.

The condition is also sufficient; for choose at pleasure a $\sigma > 0$ and a $y = y_s$ and a $z = z_t$, and let x_0 be any point of the interval (A). Then there exists by hypothesis a finite number of lines $(y = y_i, z = z_k)$, where the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ is less than the absolute value of $\sqrt{(y_s^2 + z_t^2)}$, and on each of these lines there exists a finite number of segments, each in length greater than $L > 0$, such that the projections of these segments on the line $(y = y_0, z = z_0)$ completely fill the interval (A), and such that for every point x of these segments we have

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

It follows from this that if we construct through the point x_0 a plane parallel to the YZ-plane then this plane must meet at least one of these segments, say a segment on the line $(y = \bar{y}_1, z = \bar{z}_1)$. Now choose $\bar{y}_1$ and $\bar{z}_1$ for our new y_s and z_t respectively. Then by the same reasoning as above there exists for the same σ a finite number of lines $(y = y_i, z = z_k)$ between $(\bar{y}_1, \bar{z}_1)$ and (y_0, z_0) where also the absolute value of $\sqrt{(y_i^2 + z_k^2)}$ is less than the absolute value of $\sqrt{(\bar{y}_1^2 + \bar{z}_1^2)}$, and on each of these lines there exists a finite number of segments each of length greater than L and such that their projections on the line $(y = y_0, z = z_0)$ completely fill the interval (A), such that for every point x of these segments

$$|f(x, y_i, z_k) - f(x, y_0, z_0)| < \sigma.$$

It follows from this that if we construct through the point x_0 a plane parallel to the YZ-plane then this plane must meet at least one of these segments, say a segment on the line $(y=\bar{y}_2,\ z=\bar{z}_2)$. Now choose $\bar{y}_2$ and $\bar{z}_2$ for a new y_3 and z_3 respectively and repeat the reasoning. We thus obtain a set of lines $(y=\bar{y}_i,\ z=\bar{z}_k)$ dense at $(y=y_0,\ z=z_0)$, where the absolute value of $\sqrt{(\bar{y}_i^2+\bar{z}_k^2)}$ is less than the absolute value of the same quantity for any preceding $(\bar{y}_i,\ \bar{z}_k)$, and on each of these lines there exists a finite number of segments each of length greater than L, such that the projections of these segments on the line $(y=y_0,\ z=z_0)$ completely fill the interval (A) and such that for every point x of these segments

$$|f(x,\ \bar{y}_i,\ \bar{z}_k)-f(x,\ y_0,\ z_0)|<\sigma.$$

From this we have at once by theorem 5 of this section that $f(x,\ y_0,\ z_0)$ is continuous in x at the point x_0. But x_0 is any point of the interval (A) and hence $f(x,\ y_0,\ z_0)$ is continuous in x throughout the interval (A). This proves the theorem completely.

Theorem 7.—If the function $f(x,\ y,\ z)$ is continuous in $(x,\ y)$ together and in z alone throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \quad z_0 < z \leqq z_n,$$

and if $\lim\limits_{z=z_0} f(x,\ y,\ z)=f(x,\ y,\ z_0)$ for every $(x,\ y)$ of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

then $f(x,\ y,\ z_0)$ may be discontinuous in $(x,\ y)$ together at a set of points that is everywhere dense in the region (A).

We shall prove the truth of this theorem by means of an example.

Example 1.—(Cf. Theorem 4 of section 7.)

$$\text{Let } f(x,\ y,\ z)=\sum_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x)^{\frac{2}{z}}+(\cos n!\,\pi y)^{\frac{2}{z}}\right],$$

where

$$0 \leqq x < 1, \quad 0 \leqq y < 1, \quad 0 < z \leqq 1.$$

We must show first that this example satisfies the conditions of the theorem. It is at once evident that

$$\sum_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x)^{\frac{2}{z}}+(\cos n!\,\pi y)^{\frac{2}{z}}\right]$$

is continuous in $(x,\ y)$ together for all values of z different from zero. Let

$z = c \neq 0$. We then have for every value of n and for every point (x, y) of the region

$$0 \leqq x < 1, \quad 0 \leqq y < 1 \tag{A}$$

that

$$\sum_{k=1}^{n} 1/k!\left[(\cos k!\,\pi x)^{\frac{2}{c}} + (\cos k!\,\pi y)^{\frac{2}{c}}\right] \leqq \sum_{k=1}^{n} 2/k!.$$

But $\sum_{k=1}^{\infty} 2/k!$ is a convergent series and hence $\sum_{k=1}^{\infty} 1/k!\left[(\cos k!\,\pi x)^{\frac{2}{c}} + (\cos k!\,\pi y)^{\frac{2}{c}}\right]$ also converges. Furthermore since the series $\sum_{k=1}^{\infty} 2/k!$ converges, it follows from theorem 5 of section 10 that $\sum_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x)^{\frac{2}{c}} + (\cos n!\,\pi y)^{\frac{2}{c}}\right]$, considered as a function of x, y, and n, converges uniformly throughout region (A).

Again let $x = x_0$ and $y = y_0$ where x_0 and y_0 are constant values of the given region. Then we have

$$\sum_{k=1}^{n} 1/k!\left[(\cos k!\,\pi x_0)^{\frac{2}{z}} + (\cos k!\,\pi y_0)^{\frac{2}{z}}\right] \leqq \sum_{k=1}^{n} 2/k!.$$

But $\sum_{k=1}^{\infty} 2/k!$ is a convergent series and hence

$$\sum_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x_0)^{\frac{2}{z}} + (\cos n!\,\pi y_0)^{\frac{2}{z}}\right]$$

also converges. From theorem 5 of section 10 it also follows that the series just given converges uniformly when considered as a function of n and z. Hence it is continuous in z alone. We have then shown that the series which represents the function is continuous in (x, y) together and in z alone. It still remains to show that $\lim\limits_{z=0} f(x, y, z) = f(x, y, 0)$ for every (x, y) of region (A). Since

$$\sum_{n=1}^{\infty} 1/n!\left[(\cos n!\,\pi x)^{\frac{2}{z}} + (\cos n!\,\pi y)^{\frac{2}{z}}\right],$$

when considered as a function of z, converges uniformly and is therefore continuous in z, it follows that

$$\lim_{z=0} \sum_{n=1}^{\infty} 1/n!\left[(\cos n!\pi x)^{\frac{2}{z}} + (\cos n!\pi y)^{\frac{2}{z}}\right] = \lim_{z=0} f(x, y, z) = f(x, y, 0).$$

It also follows from the uniform convergence of the above function that

$$\lim_{z=0} \sum_{n=1}^{\infty} 1/n!\left[(\cos n!\pi x)^{\frac{2}{z}} + (\cos n!\pi y)^{\frac{2}{z}}\right] = \lim_{z=0}\left[(\cos \pi x)^{\frac{2}{z}} + (\cos \pi y)^{\frac{2}{z}}\right]$$

$$+ \lim_{z=0} 1/2!\left[(\cos 2!\pi x)^{\frac{2}{z}} + (\cos 2!\pi y)^{\frac{2}{z}}\right] + \cdots\cdots$$

$$+ \lim_{z=0} 1/k!\Big[(\cos k!\pi x)^{\frac{2}{z}} + (\cos k!\pi y)^{\frac{2}{z}} + \cdots\cdots$$

Now suppose that x and y are rational, that is to say, that $x = p/q$ where p and q are integers and relatively prime to each other, and that $y = p'/q'$ where p' and q' are integers and relatively prime to each other. If then we begin with the lowest term in the series and pass along it, we shall ultimately reach a term where both q and q' are divisors of $k!$ Consider the terms of the series which precede this term. For each such term $(\cos k!\pi x)^2 < 1$ and $(\cos k!\pi y)^2 < 1$, hence $\lim\limits_{z=0} (\cos k!\pi x)^{\frac{2}{z}} = 0$ and $\lim\limits_{z=0} (\cos k!\pi y)^{\frac{2}{z}} = 0$. For the term however for which $k!$ is divisible by q and q' and for all the succeeding terms we have $(\cos k!\pi x)^2 + (\cos k!\pi y)^2 = 2$. For such values of x and y we therefore have $\lim\limits_{z=0} f(x, y, z) > 0$.

We next suppose that x and y have irrational values. Then no matter how large k is taken we always have $(\cos k!\pi x)^2 < 1$ and $(\cos k!\pi y)^2 < 1$, and hence $\lim\limits_{z=0} (\cos k!\pi x)^{\frac{2}{z}} = 0$ and $\lim\limits_{z=0} (\cos k!\pi y)^{\frac{2}{z}} = 0$. For such irrational values of x and y the series therefore always equals zero. We see then that $f(x, y, 0)$ is a function which equals zero for all irrational values of x and y, but differs from zero for all rational values of x and y. It is therefore discontinuous in (x, y) together at all rational points, that is to say at a set of points everywhere dense in the region (A).

[*To be continued.*]

In the case of this example it is easy to show that the (x, y) discontinuity cannot be greater than or equal to k (where k is some definite positive number) at every point of the set. From theorem 1 of section 7 it follows that if a function of x and y has an (x, y) discontinuity greater than or equal to k at a set of points that is everywhere dense, then it must have such an (x, y) discontinuity also at the limiting points of the set. The function $f(x, y, 0)$ of the example is a function that has an (x, y) discontinuity at a set of points that is everywhere dense. Hence if this discontinuity were greater than or equal to k at every point of the set, $f(x, y, 0)$ would have an (x, y) discontinuity greater than or equal to k at every point of the region which is impossible.

We now wish to see if the function $f(x, y, z_0)$ can be discontinuous in (x, y) together at every point of the region. It can be proved that this cannot be the case, but before doing so we need a lemma concerning a function of two variables.

Lemma.[*]—If the function $f(x, y)$ is discontinuous in (x, y) together at every point (x, y) of a region, then there always exists a sub-region within which the (x, y) discontinuity is greater than a sufficiently small quantity $A > 0$ at every point.

In order to prove this lemma we must show that the points at which $\lim\limits_{\rho=0} k < \sigma$, where $\sigma > 0$, cannot be everywhere dense in any sub-region, where k represents the oscillation of the function within any circle of radius ρ. Suppose that these points are everywhere dense in some sub-region. Now choose a set of positive quantities

$$\sigma_1 > \sigma_2 > \sigma_3 > \cdots\cdots > \sigma_n > \cdots$$

where $\lim\limits_{n=\infty} \sigma_n = 0$. Since we have assumed that the points at which $\lim\limits_{\rho=0} k < \sigma$ are everywhere dense in every sub-region, we can within any sub-region find a circle R_0 of radius $r_0 > 0$ about some point (x_0, y_0) such that at every point within this circle the oscillation k of the function is less than σ_1. Hence there exists a circle R_1 of radius r_1 greater than zero and center at (x_1, y_1) and lying wholly within the circle R_0, such that at every point within this circle R_1 the oscillation k is less than σ_2. In a like manner we can determine a circle R_2

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[*] See E. J. Townsend: Ueber den Begriff und die Anwendung des Doppellimes, 1900.

of radius $r_2 > 0$ and center (x_2, y_2) and lying wholly within the circle R_1, such that the oscillation k is less than σ_3 at every point within the circle R_2. Continue this process and we obtain in general a circle R_n of radius r_n greater than zero and center at (x_n, y_n) and lying entirely within the circle R_{n-1}, such that the oscillation k of the function is less than σ_n at every point within the circle R_n. Since now $r_1 > r_2 > r_3 > \ldots\ldots\ldots > r_n > \ldots$, we have $\lim\limits_{n=\infty} r_n = 0$, while at the same time the centers of the circles approach a definite limiting point (x_0, y_0). At this point therefore we have $\lim\limits_{\rho=0} k = 0$, where ρ is the radius of any circle about (x_0, y_0). Hence the function $f(x, y)$ is continuous in (x, y) together at this point, which contradicts the hypothesis. The lemma is therefore true.

Theorem 8.*—If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$\alpha < x < \beta, \quad a < y < b, \quad z_0 < z < z_n,$$

and if at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

we have $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$, then $f(x, y, z_0)$ cannot be discontinuous in (x, y) together at every point of the region

$$\alpha < x < \beta, \quad a < y < b, \tag{B}$$

or of any sub-region thereof.

Suppose that the limiting function $f(x, y, z_0)$ could have an (x, y) discontinuity at every point of region (B). Then there must exist, on $z = z_0$, a circle of radius $r_0 > 0$, such that at every point of this circle the (x, y) discontinuity of the function is greater than a sufficiently small quantity $A > 0$. This follows from the lemma we have just proved. Let (x_0, y_0) be the center of the above circle. Since by hypothesis $\lim\limits_{z=z_0} f(x, y, z) = f(x, y, z_0)$ at every point of the region (A) we can choose an arbitrarily small $\sigma > 0$ and can then find a $z = z_1$, such that

$$|f(x_0, y_0, z_1) - f(x_0, y_0, z_0)| < 1/2\,\sigma \tag{1}$$

where
$$z_0 < z_1 \leqq z_0 + \theta_0, \quad \theta_0 > 0.$$

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*Baire gives the corresponding theorem for a function of a single variable. See Annali di Matematica, 1899.

But the function $f(x, y, z)$ is by hypothesis continuous in (x, y) together throughout region (B) and for every z for which $f(x, y, z)$ is defined, and hence there must exist for the same σ a circle of radius r_1 about (x_0, y_0) in the plane $z = z_1$, such that for every point within this circle

$$|f(x, y, z_1) - f(x_0, y_0, z_1)| < 1/2\,\sigma \tag{2}$$

where $$z_0 < z_1 \leqq z_0 + \theta_0.$$

Let $r_1 < r_0$ so that inequalities (1) and (2) hold for every point (x, y) within this circle and we may therefore add these inequalities and thus obtain

$$|f(x, y, z_1) - f(x_0, y_0, z_0)| < \sigma. \tag{3}$$

This holds for every point (x, y) within the circle of radius r_1 about (x_0, y_0). Since by assumption $f(x, y, z_0)$ is to be discontinuous in (x, y) together at every point of region (B), there must exist in the plane $z = z_0$ within the circle of radius r_1 about (x_0, y_0) at least one point (x_1, y_1), such that

$$|f(x_1, y_1, z_0) - f(x_0, y_0, z_0)| > A. \tag{4}$$

But by hypothesis $\lim\limits_{z = z_0} f(x_1, y_1, z) = f(x_1, y_1, z_0)$ and hence there must exist a plane $z = z_2$, such that for the same σ as above

$$|f(x_1, y_1, z_2) - f(x_1, y_1, z_0)| > 1/2\,\sigma, \tag{5}$$

where $$z_0 < z_2 < z_1.$$

Since $f(x, y, z_2)$ is continuous in (x, y) together throughout region (B), there must exist for the same σ in the plane $z = z_2$ a circle of radius r_2 about (x_1, y_1), lying entirely within the circle of radius r_1, such that for every point (x, y) within this circle

$$|f(x, y, z_2) - f(x_1, y_1, z_2)| < 1/2\,\sigma. \tag{6}$$

where $$z_0 < z_2 < z_1.$$

Let $r_2 < r_1$, so that inequalities (5) and (6) hold for every point (x, y) within the circle of radius r_2. We may therefore add these inequalities and thus obtain

$$|f(x, y, z_2) - f(x_1, y_1, z_0)| < \sigma \tag{7}$$

for every point (x, y) within the circle of radius r_2. We have moreover assumed that $f(x, y, z_0)$ is discontinuous in (x, y) together at every point of region (B) and hence there must exist in the plane $z = z_0$ at least one point (x_2, y_2) within the circle of radius r_2 about (x_1, y_1) for which

$$|f(x_2, y_2, z_0) - f(x_1, y_1, z_0)| > A. \tag{8}$$

We can continue this reasoning indefinitely and have, in general, that there exists a circle of radius r_n about (x_{n-1}, y_{n-1}) and lying wholly within each of the preceding circles and that there also exists a corresponding plane $z = z_n$, where $z_1 > z_2 > z_3 > \ldots > z_n > \ldots > z_0$, such that for every point (x, y) within this circle the following two inequalities hold

$$|f(x, y, z_n) - f(x_{n-1}, y_{n-1}, z_0)| < \sigma \tag{9}$$

$$|f(x_{n-1}, y_{n-1}, z_0) - f(x_{n-2}, y_{n-2}, z_0)| > A. \tag{10}$$

Now as n increases indefinitely, the radius r_n decreases indefinitely and approaches zero as limit. The centers (x_n, y_n) of the circles must at the same time approach a definite fixed limiting point $(\bar{x}, \bar{y})$. We thus obtain the following inequalities

$$|f(x_0, y_0, z_0) - f(\bar{x}, \bar{y}, z_1)| < \sigma \qquad \text{(from 3)}$$

$$|f(\bar{x}, \bar{y}, z_2) - f(x_1, y_1, z_0)| < \sigma . \qquad \text{(from 7)}$$

$$|f(x_1, y_1, z_0) - f(x_0, y_0, z_0)| > A .$$

If now we choose $0 < \sigma < 1/4\,A$ and combine these three inequalities we obtain

$$|f(\bar{x}, \bar{y}, z_2) - f(\bar{x}, \bar{y}, z_1)| > 1/2\,A.$$

In a similar manner we obtain the following inequalities

$$|f(\bar{x}, \bar{y}, z_3) - f(\bar{x}, \bar{y}, z_2)| > 1/2\,A$$

$$|f(\bar{x}, \bar{y}, z_4) - f(\bar{x}, \bar{y}, z_3)| > 1/2\,A$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$|f(\bar{x}, \bar{y}, z_n) - f(\bar{x}, \bar{y}, z_{n-1})| > 1/2\,A.$$

But the z's are dense at $z = z_0$ and hence we obtain from the last inequality

$$\lim_{\substack{z_{n-1} = z_0 \\ z_n = z_0}} [f(\bar{x}, \bar{y}, z_n) - f(\bar{x}, \bar{y}, z_{n-1})] > 1/2\,A \neq 0.$$

and hence the limit $\lim\limits_{z = z_0} f(x, y, z)$ does not exist at the point $(\bar{x}, \bar{y})$. The assumption that $f(x, y, z_0)$ has an (x, y) discontinuity at every point of region (B) is therefore false and the theorem is true.

Remark 1.—In the above theorem we assumed that $\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$ for every point of region (A), but we were only able to prove that the limiting function $f(x, y, z_0)$ is not discontinuous in (x, y) together at every point of region (B). As a matter of fact, $f(x, y, z_0)$ may have an (x, y) discontinuity at every point of the boundaries of region (A). We show this by means of the following example.

Example 2. Let $f(x, y, z) = \dfrac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1}$,

where $0 \leqq x \leqq 10$, $0 \leqq y \leqq 10$, $0 < z \leqq 10$, and $\lim\limits_{z=0} f(x, y, z) = f(x, y, 0)$.

The value of the limit $\lim\limits_{z=0} f(x, y, z)$ is evidently zero for all values of x and y which make xy integral, for in this case $\sin \pi xy = 0$. But if we let $2n < xy < 2n + 1$, where n is some integer, then for all such points we have $0 < \sin \pi xy < 1$. Hence for all such values of x and y

$$\lim_{z=0} \frac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1} = \lim_{z=0} \frac{1 - \dfrac{1}{(1 + \sin \pi xy)^{1/z}}}{1 + \dfrac{1}{(1 + \sin \pi xy)^{1/z}}} = +1 .$$

On the other hand if we let $2n + 1 < xy < 2n + 2$, where n is a positive integer, then we obtain

$$\lim_{z=0} \frac{(1 + \sin \pi xy)^{1/z} - 1}{(1 + \sin \pi xy)^{1/z} + 1} = \frac{0 - 1}{0 + 1} = -1 .$$

But when either x or y is equal to zero, then $f(x, y, 0) = 0$ along the whole X-axis and the whole Y-axis. Hence $f(x, y, 0)$ is discontinuous in (x, y) together at every point of these lines.

Remark 2—In theorem 7 we showed that, for the example there given, the (x, y) discontinuity could not be greater than or equal to k at every point. We are now in a position to prove in general that, if the function $f(x, y, z)$ satisfies the conditions of theorem 7, then the limiting function $f(x, y, z_0)$ cannot have an (x, y) discontinuity greater than or equal to k at a set of points that is everywhere dense. For if this were possible, then it follows from theorem 1 of section 7 that the function $f(x, y, z_0)$ has an (x, y) discontinuity greater than or equal to k also at the limiting points, that is to say it would have such an (x, y) discontinuity at every point of the region. This is however impossible by theorem 8 just proved.

Theorem 9.—If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha < x < \beta, \quad y_0 < y < y_n, \quad z_0 < z < z_n,$$

and if for every point of the interval $\alpha \leqq x \leqq \beta$, $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$, then $f(x, y_0, z_0)$ cannot be discontinuous in x alone at every point of the interval $\alpha < x < \beta$.

The proof of this theorem is strictly analogous to that of theorem 8. We replace the circles by intervals about x_0 on lines parallel to the X-axis, in a manner similar to that employed in theorem 6.

Theorem 10.—If the function $f(x, y, z)$ is continuous in (x, y) together and in z alone throughout the region

$$\alpha < x < \beta, \; a < y < b, \; z_0 < z < z_n,$$

and if at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \; a \leqq y \leqq b, \tag{A}$$

$\lim\limits_{z = z_0} f(x, y, z) = f(x, y, z_0)$, then the function $f(x, y, z_0)$ must be continuous in (x, y) together at a set of points that is at least dense everywhere in the region

$$\alpha < x < \beta, \; a < y < b. \tag{B}$$

Suppose that this is not so. Then there must exist in the plane $z = z_0$ some circle R_0 of radius r_0 greater than zero and lying entirely within region (B), such that within this circle there is no point for which $\lim\limits_{\rho = 0} k = 0$, where ρ is the radius of any variable circle about the point and k is the oscillation of the function within this circle. Within the circle R_0 therefore $\lim\limits_{\rho = 0} k \neq 0$ for every point. Hence every point (x, y) within the circle R_0 is a point at which $f(x, y, z_0)$ has an (x, y) discontinuity. This however contradicts theorem 8, and hence the theorem is true.

This theorem may be extended in a manner similar to that employed in the extension of theorem 5 of section 7. We thus obtain the following corollary.

Corollary.—If the function $f(x, y, z)$ satisfies the conditions of theorem 10, then the points at which $f(x, y, z_0)$ is continuous in (x, y) together form a set of the second category, and hence a non-enumerable set.

Theorem 11.—If the function $f(x, y, z)$ is continuous in x alone and in (y, z) together throughout the region

$$\alpha < x < \beta, \quad y_0 < y < y_m, \quad z_0 < z < z_n,$$

and if $\lim\limits_{\substack{y = y_0 \\ z = z_0}} f(x, y, z) = f(x, y_0, z_0)$ for every point of the interval $\alpha \leqq x \leqq \beta$, then $f(x, y_0, z_0)$ is continuous in x alone at a set of points that is at least everywhere dense in the interval $\alpha < x < \beta$. Moreover this set belongs to the second category and is non-enumerable.

Suppose that this is not true. Then there must exist on the X-axis some interval greater than zero and lying wholly within the interval $\alpha < x < \beta$, such that within this interval there is no point x for which the limit of the oscillation of the function $f(x, y_0, z_0)$ is zero. Within this interval therefore the function $f(x, y_0, z_0)$ has an x discontinuity at every point. But this contradicts theorem 9, and hence our theorem is true as regards the first part.

The second part of the theorem is proved in exactly the same manner as was the corollary of the preceding theorem.

CHAPTER III.

SERIES OF FUNCTIONS OF TWO REAL VARIABLES.

10. *Properties of Series of Functions of Two Real Variables.*—Suppose that we have given

$$S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots\ldots + u_n(x, y),$$

where each u is finite and a continuous function of (x, y) together throughout a certain region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b,$$

for every value of n $(n = 1, 2, 3, \ldots\ldots)$. Since each u is finite and continuous in (x, y) together and since n is also finite, it follows that $S_n(x, y)$ must be finite and continuous in (x, y) together throughout the given region. If then for every constant value of x and y of the region, $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$, then $S(x, y)$ represents a function of (x, y) throughout the given region, and the question now arises as to the properties of this limiting function $S(x, y)$.

If we wish, we may consider $S_n(x, y)$ also as a function of the three variables x, y, and n. This enables us to apply the results of the preceding chapters to the present problem. In order to do so, we define x and y as above, namely as any values of

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b.$$

In order to define z we put $z = 1/n$, so that the z of the preceding chapters takes on the values $z = 1, 1/2, 1/3, \ldots, 1/n \ldots$, which is a set of values dense at $z = 0$. We then have

$$S(x, y) = \lim_{n=\infty} S_n(x, y) = \lim_{z=0} f(x, y, z),$$

where $f(x, y, z) = S_n(x, y)$ for every point (x, y) of the given region. It follows from this that everything which was shown to be true for the limiting function $f(x, y, z_0)$ in the case of a function of three variables must also hold for the series as defined above. The converse of this is, however, not true, for if we have proved a theorem about an infinite series, it does not follow necessarily that an analogous theorem exists for a function of three variables. The reason for this is to be found in the fact that, in the case of a function of three variables, z is defined for all positive values, while in the case of the series it is defined only for a set of values dense at $z = 0$. It may therefore happen that a theorem holds for a set of values dense at $z = z_0$, where $z_0 < z \leqq z_0 + \theta_0$, while the theorem will not hold for *all* values z of the interval $z_0 < z \leqq z_0 + \theta_0$.

From what has been said follows that to the three different kinds of uniform convergence of $f(x, y, z)$ to its limiting function, as they were defined in the last chapter, there correspond three different kinds of uniform convergence of $S_n(x, y)$ to $S(x, y)$. These are as follows:

(1) If $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$ is, for every finite value of n, finite and continuous in (x, y) together throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if for every point (x, y) of this region $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$, then $S_n(x, y)$

converges uniformly to $S(x, y)$ throughout region (A), if having chosen an arbitrarily small $\sigma > 0$, we can then find a positive integer m, such that

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma,$$

when $n > m$ and for all points (x, y) of region (A).

(2) If $S_n(x, y)$ and $S(x, y)$ are defined as in (1), then $S_n(x, y)$ converges "einfach gleichmässig" throughout region (A) to $S(x, y)$, if having chosen an arbitrarily small $\sigma > 0$ we can then find a positive integer m, such that

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma,$$

for some set of integral values n greater than m, which however increase without limit. In this case the n's do not assume all integral values.

(3) If $S_n(x, y)$ and $S(x, y)$ are defined as in (1), then $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout the region (A), if having chosen an arbitrarily small $\sigma > 0$, there then exists a plane $z = 1/n$ and between this plane $z = 1/n$ and the plane $z = 0$ a finite number of planes, and on each of these a finite number of rectangles, having their sides parallel to the coordinate axes, such that their projections on the plane $z = 0$ completely fill the region (A) and that for every point (x, y) of these rectangles

$$|R_n(x, y)| \equiv |S_n(x, y) - S(x, y)| < \sigma.$$

We will now give the theorems that deal with the properties of a series of functions of two real variables. Unless something is said to the contrary, we shall in every case make the following assumptions about the series:

(1) $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y),$

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \qquad \text{(A)}$$

where $u_i(x, y)$, $(i = 1, 2, 3, \ldots, n)$ is continuous in (x, y) together throughout region (A);

(2) $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$, for every constant pair of values (x, y) of region (A).

From theorem 1 of section 8 we have immediately

Theorem 1.—The necessary and sufficient condition that $S_n(x, y)$ shall converge uniformly to $S(x, y)$ throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \qquad \text{(A)}$$

is that at every point (x_0, y_0) of this region, boundaries included, the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ exists and equals $S(x_0, y_0)$.

Example 1.

$$\text{Let } S_n(x, y) = \frac{xy}{xy+1} + \left[\frac{2xy}{4xy+1} - \frac{xy}{xy+1}\right] + \cdots\cdots + \left[\frac{nxy}{n^2xy+1} - \frac{(n-1)xy}{(n-1)^2xy+1}\right] = \frac{nxy}{n^2xy+1}$$

where $\qquad 0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1.$

In this case the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ exists and equals zero at every point (x, y) of the region, and hence $S_n(x, y)$ converges uniformly to $S(x, y)$ throughout the region.

Example 2.

$$\text{Let } S_n(x, y) = \frac{xy}{x^3+y^3+1} + \left[\frac{4xy}{8x^3+8y^3+1} - \frac{xy}{x^3+y^3+1}\right] + \cdots\cdots + \left[\frac{n^2xy}{n^3x^3+n^3y^3+1} - \frac{(n-1)^2xy}{(n-1)^3x^3+(n-1)^3y^3+1}\right] = \frac{n^2xy}{n^3x^3+n^3y^3+1}$$

where $\qquad 0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1.$

Here the simultaneous limit $\lim\limits_{\substack{x=x_0\\ y=y_0\\ n=\infty}} S_n(x, y)$ does not exist for $x = y = 0$ and hence $S_n(x, y)$ does not converge uniformly to $S(x, y)$.

From theorem 1 of section 9 follows:

Theorem 2.—If $S_n(x, y)$ converges uniformly to $S(x, y)$ throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b,$$

then $S(x, y)$ is continuous in (x, y) together throughout the given region.

This condition is only a sufficient condition for the continuity of $S(x, y)$ in (x, y) together. Thus in example 2 the function $S(x, y)$ is continuous in (x, y) together throughout the region given, but $S_n(x, y)$ does not converge uniformly to $S(x, y)$.

The necessary and sufficient condition for the continuity of $S(x, y)$ in (x, y) together at some fixed point (x_0, y_0) of the region follows from theorem 3 of section 9. It may be stated as follows:

Theorem 3.—The necessary and sufficient condition that the function $S(x, y)$ is continuous in (x, y) together at a fixed point (x_0, y_0) of the given region, is that having chosen an arbitrarily small $\sigma > 0$ and having some set of positive integers $m_1, m_2, m_3, \ldots, m_k, \ldots$, which increase indefinitely, there shall then exist about (x_0, y_0) corresponding to each m_k a circle of radius $r_k > 0$ (which may vary with m_k), such that for all points (x, y) within this circle

$$|S_{m_k}(x, y) - S(x, y)| < \sigma.$$

From theorem 5 of section 9 we obtain the necessary and sufficient condition that $S(x, y)$ is continuous in (x, y) together throughout the given region. The condition may be stated as follows:

Theorem 4.—The necessary and sufficient condition that $S(x, y)$ is continuous in (x, y) together at every point (x, y) of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b. \tag{A}$$

is that for every arbitrarily small $\sigma > 0$ there shall exist a plane $z = 1/n$ and between this plane and the plane $z = 0$ a finite set of planes parallel to the XY-plane, each of these to contain a finite set of rectangles whose sides are parallel to the coordinate axes and whose projections on the plane $z = 0$ completely fill the region (A), such that for every point (x, y) of these rectangles

$$|S_n(x, y) - S(x, y)| < \sigma.$$

Theorem 5.—If the sum of the maximum values of the terms of a series $S_n(x, y)$ converges absolutely within a given region, then the series converges also uniformly.

The following conditions are given:

(1) $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y)$

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b. \tag{A}$$

Here as elsewhere we confine ourselves to the case where $u_i\,(x, y)$ is continuous in (x, y) together;

(2) $\lim\limits_{n=\infty} S_n\,(x, y) = S\,(x, y)$ for every pair of values (x, y) of region (A);

(3) $|u_i\,(x, y)| = M_i$, where M_i is positive and finite, and this is to hold for every (x, y) of region (A);

(4) $\sum\limits_{i=1}^{\infty} M_i$ converges.

We are to show that under these conditions $S_n\,(x, y)$ converges uniformly throughout region (A).

Since $\sum\limits_{i=1}^{\infty} M_i$ is convergent, we can choose an arbitrarily small $\varepsilon > 0$ and can then find an m, such that

$$(M_{m+1} + M_{m+2} + \ldots\ldots + M_{m+p}) < \varepsilon,\ p = 1, 2, 3, \ldots\ldots$$

Hence for m sufficiently large

$$|u_{m+1}\,(x, y) + u_{m+2}\,(x, y) + \ldots\ldots + u_{m+p}\,(x, y)| < \varepsilon,\ p = 1,\ 2, \ldots\ldots$$

for every point (x, y) of region (A). We therefore have at every point of this region

$$|\,G - S_n\,(x, y)\,| < \varepsilon, \text{ where } n > m,$$

and hence the simultaneous limit $\lim\limits_{\substack{x = x_0 \\ y = y_0 \\ n = \infty}} S_n\,(x, y)$ exists at every point of region (A), boundaries included.

From 4 of the hypothesis follows that $S_n\,(x, y)$ also converges and hence by virtue of theorem 3 of section 2

$$\lim_{\substack{x = x_0 \\ y = y_0 \\ n = \infty}} S_n\,(x, y) = \lim_{n = \infty}\ \lim_{\substack{x = x_0 \\ y = y_0}} S_n\,(x, y).$$

But $S_n\,(x, y)$ is continuous in (x, y) together and hence

$$\lim_{n = \infty}\ \lim_{\substack{x = x_0 \\ y = y_0}} S_n\,(x, y) = \lim_{n = \infty} S_n\,(x, y) = S\,(x_0, y_0)$$

at every point (x_0, y_0) of region (A). The simultaneous limit therefore not only exists, but it also equals the value of the function at the point, and hence the series converges uniformly by theorem 1 of this section.

In theorem 3 we have given the necessary and sufficient condition that the limiting function $S(x, y)$ is continuous in (x, y) together. The question now presents itself as to whether the limiting function may be discontinuous and, if it is discontinuous, how often it may be discontinuous. We obtain an answer to this question from theorem 7 of section 9. It is as follows:

Theorem 6.—If $S_n(x, y)$ and $S(x, y)$ are defined as in theorem 1, then the limiting function $S(x, y)$ may be discontinuous in (x, y) together at a set of points that is everywhere dense in the given region, but in this case the (x, y) discontinuities cannot all be greater than or equal to some finite positive number $\bar{k}$.

The proof follows immediately from the example under theorem 7 of section 9. If in this we let $z = 1/n$, then the example becomes

$$S_n(x, y) = \sum_{k=1}^{n} 1/k! \left[(\cos k!\,\pi x)^{2n} + (\cos k!\,\pi y)^{2n} \right],$$

$$0 \leqq x < 1, \quad 0 \leqq y < 1.$$

From theorem 8 of section 9 we obtain

Theorem 7.—If $S_n(x, y)$ and $S(x, y)$ are defined as before, then the limiting function $S(x, y)$ cannot be discontinuous in (x, y) together at every point of the region $\alpha < x < \beta$, $a < y < b$.

From theorem 10 of section 9 we have

Theorem 8.—If $S_n(x, y)$ and $S(x, y)$ are defined as before, then $S(x, y)$ must be continuous in (x, y) together at a set of points that is at least everywhere dense in the region $\alpha < x < \beta$, $a < y < b$. Moreover this set of points is a set of the second category and hence is non-enumerable.

11. *The Integration of Series Term by Term.*

In this section we shall consider the following question. Suppose we know:

(1) that $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y)$, and that the simultaneous integral $\int_\alpha^\beta \int_a^b u_k(x, y)\, dxdy$ exists for every finite value of k. Then the simultaneous integral $\int_\alpha^\beta \int_a^b S_k(x, y)\, dxdy$ also exists for every finite value of k, where our region is defined as

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b. \tag{A}$$

(2) We have also $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every constant pair of values (x, y) of region (A).

This being the case the question arises under what conditions will the integral of $S(x, y)$ exist over the region, and if it exists, under what conditions will it equal the term by term integral? This is to say, under what conditions do we have

$$\int_a^\beta \int_a^b \lim_{n=\infty} S_n(x, y)\, dxdy = \lim_{n=\infty} \int_a^\beta \int_a^b S_n(x, y)\, dxdy ?$$

Before answering these questions, let us first explain what we mean by the simultaneous integral. In dealing with a function of two variables we meet with two kinds of integrals, the twice-taken or double integral and the simultaneous integral.

The twice-taken or double integral is defined as the result of integrating the function $f(x, y)$ first with respect to one of the variables, say x, between the given limits (α, β), and then integrating the resulting function with respect to y between the limits (a, b). This gives us the integral which is denoted by $\int_a^b dy \int_a^\beta f(x, y)\, dx$. The integral in question is equivalent to a double limit.

The simultaneous integral $\int_a^\beta \int_a^b f(x, y)\, dxdy$ may be defined as follows: Divide the region into a set of rectangles by means of lines parallel to the X and Y axes. Then multiply the area of each rectangle by the value of the function at some point of the corresponding rectangle, and add the results. We then have

$$\begin{aligned}\Sigma = (x_0 - \alpha)(y_0 - a) f(\bar{x}_0, \bar{y}_0) + (x_1 - x_0)(y_0 - a) f(\bar{x}_1, \bar{y}_0) + \dots\dots\\ + (x_i - x_{i-1})(y_k - y_{k-1}) f(\bar{x}_i, \bar{y}_k) + \dots + (\beta - x_{l-1})(b - y_{m-1}) f(\bar{x}_l, \bar{y}_m)\end{aligned}$$

where $f(\bar{x}_i, \bar{y}_k)$ represents the value of the function at some point of the corresponding rectangle. Now take the limit of this sum as l and m increase indefinitely, while at the same time the dimensions of the rectangles decrease indefinitely. We then define the simultaneous integral as the limit thus obtained. The simultaneous integral is therefore equivalent to a simultaneous limit.

In the following pages we shall consider simultaneous integrals as thus defined. We must now answer the following question. If the simultaneous integral $\int_a^\beta \int_a^b S_n(x, y)\, dxdy$ exists, what is the necessary and sufficient condition that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ shall also exist? This condition is given in the following theorem.

Theorem 1.[*]—If (1) $S_n(x, y) = \sum_{k=1}^{n} u_k(x, y)$, for all values x and y of the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b; \tag{A}$$

(2) $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every constant pair of values (x, y) of region (A), where both $S_n(x, y)$ and $S(x, y)$ are finite;

(3) the simultaneous integral $\int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ exists for every constant value of n;

then the necessary and sufficient condition that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ shall exist is that, having chosen an arbitrarily small $\varepsilon > 0$ and $\sigma > 0$, independent of each other, and having also chosen an $n = m_1$ we can then always find an $m_2 > m_1$, such that for all values (x, y) of region (A), excepting at most for those contained within a set of rectangles finite in number and of total area less than ε, and for some number m which may vary with (x, y) but where always $m_1 < m < m_2$, we have

$$|R_m(x, y)| \equiv |S(x, y) - S_m(x, y)| < \sigma .$$

This condition may be expressed otherwise as follows: Choose as coordinate axes the axes X, Y, and $1/n$ and choose ε and σ as above. We must then be able to find a plane $1/n = m_s$ and between this plane and the plane $1/n = 0$ a finite number of planes, $1/n = m_1, m_2, m_3, \ldots, m_p$, such that after having cut out from these within the given region a finite number of rectangles of total area less than ε, these rectangles to have their sides parallel to the coordinate axes X and Y, the function $S_n(x, y)$ shall converge uniformly by areas to $S(x, y)$ throughout the rest of the region (A). It follows at once from theorem 4 of section 1 that $S(x, y)$ must be continuous in (x, y) together throughout the rest of the region. If the above conditions are fulfilled, then we say with Arzela that $S_n(x, y)$ "converges uniformly by areas in general" to $S(x, y)$.

The condition of the theorem is necessary. We suppose then that the simultaneous integral $\int_a^\beta \int_a^b S(x, y)\, dxdy$ exists over the region. The set of points at which $S(x, y)$ has an (x, y) discontinuity greater than or equal to σ, can therefore be at most a discrete set, that is to say, these points must be such that they can be enclosed in a set of rectangles finite in number and of total area less than ε. Let us denote this set by $t_1^{(0)}, t_2^{(0)}, t_3^{(0)}, \ldots, t_p^{(0)}$. Since now the simultaneous integral $\int_a^\beta \int_a^b S_n(x, y)\, dxdy$ also exists on each of the planes $1/n = m_1, m_2, \ldots$ the points at which $S_{m_k}(x, y)$, $(k = 1, 2, 3, \ldots)$, has an (xy) discontinuity greater than or equal to σ can form at most a discrete set, and hence on each of the planes $1/n = m_1, m_2, m_3, \ldots$, this set of points can be enclosed in a finite set of rectangles of total area less than ε. Denote this set of rectangles by $t_1^{(k)}, t_2^{(k)}, t_3^{(k)}, \ldots, t_q^{(k)}$, for any plane $1/n = m_{m_k}$. Now let $ABCD$ be one of the rectangles $t^{(1)}$ in the plane $1/n = m_1$ and let its projection on the plane $1/n = 0$ be $A'B'C'D'$. See figure 13.

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[*] See Arzela, "Sulle serie di funzioni". R. Accademia dell 'Instituto di Bologna, 1900.

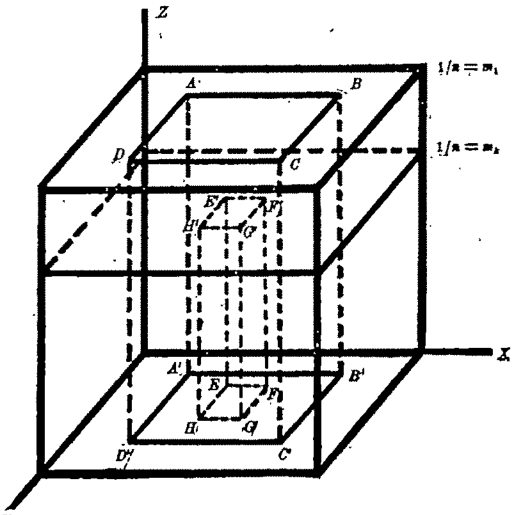

FIG. 13.

We can now assert that if $A'B'C'D'$ does not contain any of the rectangles $t^{(0)}$ or a part of such a rectangle, then as we pass from $1/n = m_1$ to $1/n = m_2$, to $1/n = m_3$, etc., we shall ultimately reach a plane $1/n = m_k$, such that the projections of $ABCD$ on $1/n = m_k$ and on all planes between this and the plane $1/n = 0$, do not contain a rectangle $t^{(k)}$ or a part of such a rectangle. In other words, we can find a number $1/n = m_k$ such that for this and all values between this and $1/n = 0$ the function $S_k(x, y)$ is continuous in (x, y) together throughout the corresponding rectangle. Suppose namely that this is not the case. Then as we pass through the values $1/n = m_1, m_2, m_3, \ldots,$ we always have a rectangle in every plane or at least in a set of planes dense at $1/n = 0$, such that this rectangle lies within the projection of $ABCD$ on the plane in question and furthermore such that within this rectangle the function $S_n(x, y)$ has (x, y) discontinuities greater than or equal to σ. Let us denote the set of rectangles we thus obtain by $T_1, T_2, T_3, \ldots, T_k, \ldots$. Two cases may now arise: either T_k has a lower limit T greater than zero or it has the lower limit zero as k increases indefinitely.

In the first case let $T = EFGH$ (see figure 13). Then no matter how large we take k, we always have in the plane $1/n = m_k$ a rectangle $E'F'G'H'$, corresponding to $EFGH$, such that within this rectangle the function $S_n(x, y)$ has (x, y) discontinuities greater than or equal to σ. Let (x_0, y_0) be a point of this rectangle at which the (x, y) discontinuity is greater than or equal to σ. Then we have for all values of n and for some point (x, y) of this rectangle

$$|S_n(x, y) - S_n(x_0, y_0)| > \sigma. \tag{1}$$

Since by hypothesis $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$, we have also

$$|S(x, y) - S_n(x, y)| < 1/4\sigma; \quad n > m_k. \tag{2}$$

Furthermore since $S(x, y)$ is continuous in (x, y) together throughout the rectangle $EFGH$, we have

$$|S(x_0, y_0) - S(x, y)| < 1/4\,\sigma. \tag{3}$$

Combining these three inequalities we obtain

$$|S_n(x_0, y_0) - S(x_0, y_0)| > 1/4\,\sigma; \quad n > m_k. \tag{4}$$

Inequality 4 however contradicts the hypothesis that $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every pair of values (x, y) of region (A), and hence the assumption we made is false. It therefore follows that, if we have in each of a set of planes dense at $1/n = 0$ a rectangle of lower limit $T \neq 0$, within which the function $S_n(x, y)$ has an (x, y) discontinuity greater than or equal to σ, then the function $S(x, y)$ has such an (x, y) discontinuity within the rectangle T in the plane $1/n = 0$.

Suppose in the second place that the limit of T is zero. We have then in the plane $1/n = m_s$ a rectangle T_s within which any (x, y) discontinuity of

$S_n(x, y)$ is greater than or equal to σ. Furthermore as s increases indefinitely the area of this rectangle approaches zero. Let this occur at the point (x', y') and let (x'', y'') be any other point of the rectangle T_s. If now we drop a perpendicular from (x'', y') to the plane $1/n = 0$, then one of two things will happen; either there exists a plane $1/n = m_k$, such that the perpendicular does not meet a rectangle $t^{(i)}$ in any plane $1/n = m_i$ between $1/n = m_k$ and $1/n = 0$, or no matter what plane $1/n = m_k$ we choose, the perpendicular will always meet a rectangle $t^{(k)}$. We must now show that in the latter case the perpendicular also meets one of the rectangles $t^{(0)}$ in the plane $1/n = 0$. We assume that the perpendicular meets a rectangle $t^{(k)}$ in every plane $1/n = m_k$. Then for every point of this rectangle

$$|S_{m_k}(x, y) - S_{m_k}(x'', y'')| > \sigma. \tag{1}$$

Since moreover $\lim_{n=\infty} S_n(x, y) = S(x, y)$ for every point of region (A), it follows that

$$|S(x, y) - S_{m_k}(x, y)| < 1/4\,\sigma \tag{2}$$

for every m_k greater than some m'. Likewise

$$|S_{m_k}(x'', y'') - S(x'', y'')| < 1/4\,\sigma \tag{3}$$

for every m_k greater than some m''. Let $m'' > m'$; then for every m_k greater than m'', (1), (2), and (3), may be added and hence

$$|S(x, y) - S(x'', y'')| > 1/4\,\sigma \tag{4}$$

for every (x, y) of some rectangle about (x'', y'') in the plane $1/n = 0$. Hence $S(x, y)$ is discontinuous in (x, y) together at (x'', y'') and the perpendicular from (x'', y'') therefore must meet one of the rectangles $t^{(0)}$ in the plane $1/n = 0$. In a like manner we can prove that the perpendicular from (x', y') meets one of the rectangles $t^{(0)}$ in the plane $1/n = 0$.

We are now ready to complete the proof of the original theorem. In order to do so, let us cut out from the plane $1/n = 0$ all the rectangles $t^{(0)}$, that contain the points at which $S(x, y)$ has an (x, y) discontinuity greater than or equal to σ. As has been shown, this set of rectangles is finite in number and of total area less than ε. Lines through the vertices of these rectangles parallel to the X and Y axes divide the remaining part of region (A) into a finite number of rectangles. Number these $1, 2, 3, \ldots, q$. Now consider rectangle number 1. From what has been proved it follows that there must exist a plane $1/n = m_k$ such that for this plane and for all planes between this and the plane $1/n = 0$

the function $S_n(x, y)$ is continuous in (x, y) together throughout the rectangle in question. Hence by theorem 4 of section 10 the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout this rectangle. In a like manner we prove that the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout each of the remaining rectangles $2, 3, \ldots, q$. We have thus shown that, after we have cut out from region (A) a finite number of rectangles of total area less than ε, the function $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout the rest of region (A). This proves that the condition of the theorem is necessary.

The condition is also sufficient. By hypothesis then $S_n(x, y)$ converges uniformly by areas to $S(x, y)$ throughout region (A), excepting at the points of a finite number of rectangles of total area less than ε. We can therefore choose at pleasure a plane $1/n = m_\varepsilon$ and there will then exist between this plane and the plane $1/n = 0$ a finite number of planes having the following properties. From these planes, including the plane $1/n = 0$, are cut out a finite number of rectangles, $t_1, t_2, t_3, \ldots, t_p$, of total area less than ε. These rectangles must be such that, from some m_ε on, their projections on the plane $1/n = 0$ cannot completely fill region (A), otherwise the function $S(x, y)$ would be totally discontinuous in (x, y) together throughout region (A). In the remaining part of each plane there exists a finite number of adjacent rectangles (each of area greater than $G > 0$) whose projections on the plane $1/n = 0$ completely fill the remaining part of region (A), such that for every point (x, y) of these rectangles

$$|S_k(x, y) - S(x, y)| < \sigma. \tag{1}$$

Let us denote $S_k(x, y)$ for the points of these rectangles by $K(x, y)$. Then the following inequality always is true

$$|K(x, y) - S(x, y)| < \sigma. \tag{2}$$

Moreover the simultaneous integral $\int_\alpha^\beta \int_a^b K(x, y)\, dxdy$ exists throughout the region for which $K(x, y)$ is defined. Now consider the identity

$$S(x, y) \equiv K(x, y) + S(x, y) - K(x, y),$$

which holds for all points (x, y) for which $K(x, y)$ is defined. In the right member of this identity $K(x, y)$ may have an (x, y) discontinuity greater than or equal to σ only at a discrete set of points. The function $[S(x, y) - K(x, y)]$ can have such an (x, y) discontinuity only at points of the rectangles

$$t_1, t_2, t_3, \ldots, t_p.$$

It follows from this that the sum $K(x, y) + S(x, y) - K(x, y)$ can have such an (x, y) discontinuity only at a discrete set of points. This being the case, it is at once evident that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ exists.

The theorem just proved holds also in the general case of a function of three variables, the proof being similar to that just given.

Having thus obtained the necessary and sufficient condition that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ exists, the next problem is the following.

If the simultaneous integral

$$\int_\alpha^\beta \int_a^b \lim_{n=\infty} S_n(x, y)\, dxdy = \int_\alpha^\beta \int_a^b S(x, y)\, dxdy = \theta(x, y)$$

exists for every point (x, y) of region (A) and if it is finite and continuous, under what conditions will the simultaneous integral $\lim\limits_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ exist and equal $\theta(x, y)$?

We have here two cases to consider. For every fixed value of n but for a variable pair of values (x, y) the function $|S_n(x, y)|$ has an upper limit $G > 0$. This G is a function of n, if we consider n as variable. Then we have the following two cases:

(1) G has an upper finite limit G' as n increases indefinitely;

(2) G has no finite upper limit as n increases indefinitely.

These two cases give rise to different conditions which are stated in the following theorems. Arzela* has proved the analogous theorems for a series of functions of a single variable. The proofs given below are generalizations of Arzelà's proofs.

Theorem 2.—If $S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \dots + u_n(x, y)$ where each of the simultaneous integrals $\int_\alpha^\beta \int_a^b u_k(x, y)\, dxdy$ exists throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if also $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every (x, y) of region (A), and if for every constant value of n and for every (x, y) of region (A) we have $S_n(x, y) < G$, where G is finite, then the necessary and sufficient condition that the simultaneous integral $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ shall exist at every point of region (A) and that at the same time

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* Cf. Arzela: "Sulle serie di funzioni"; R. Accademia dell' Instituto di Bologna, 1900.

$$\int_\alpha^\beta \int_a^b S(x, y)\, dxdy = \lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$$

is that $S(x, y)$ shall converge uniformly by areas in general to $S(x, y)$ throughout region (A).

That the condition is necessary follows immediately from theorem 1 of this section.

The condition is also sufficient. By hypothesis therefore $S_n(x, y) < G$ for every constant n and for every (x, y) of region (A). Hence, since $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$

$$|S(x, y) - S_n(x, y)| < 2G \qquad (1)$$

so that $S(x, y)$ also has a finite upper limit. Let us now choose independently of each other an arbitrarily small $\sigma < 0$ and $\varepsilon < 0$. Since $S_n(x, y)$ converges uniformly by areas in general to $S(x, y)$ throughout region (A), we can find an s such that the points at which

$$|S(x, y) - S_{s+p}(x, y)| \geqq \sigma, \quad (p = 1, 2, 3, \ldots .) \qquad (2)$$

form a discrete set, that is they can be enclosed in a finite number of rectangles of total area less than ε. Consider now the function $[S(x, y) - S_{s+p}(x, y)]$. This function can evidently have an (x, y) discontinuity greater than or equal to $1/2\sigma$ only at points (x, y) for which (2) holds, that is at a discrete set of points, and this is true for all values of p. Hence the simultaneous integral $\int_\alpha^\beta \int_a^b [S_{s+p}(x, y) - S(x, y)]\, dxdy$ exists. Moreover the value of this integral is less than or at most equal to $(x - a)(y - a)\sigma + 2G\varepsilon$. Since by hypothesis the simultaneous integrals $\int_\alpha^\beta \int_a^b S_{s+p}(x, y)\, dxdy$ and $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ also exist for any (x, y) of the region, we have

$$\int_\alpha^\beta \int_a^b S_{s+p}(x, y)\, dxdy - \int_\alpha^\beta \int_a^b S(x, y)\, dxdy = \int_\alpha^\beta \int_a^b [S_{s+p}(x, y) - S(x, y)]\, dxdy < (x - a)(y - a)\sigma + 2G\varepsilon < \varepsilon'.$$

Hence we obtain in the limit $\lim\limits_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy = \int_\alpha^\beta \int_a^b S(x, y)\, dxdy$, which was to be proved.

Before taking up the case where G has no upper finite limit, we must first prove a couple of lemmas. These theorems were first proved by Scheeffer* for a function of single variable.

Lemma 1.—If $F(x, y)$ and $f(x, y)$ are two functions of x and y which are continuous in (x, y) together throughout the region

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

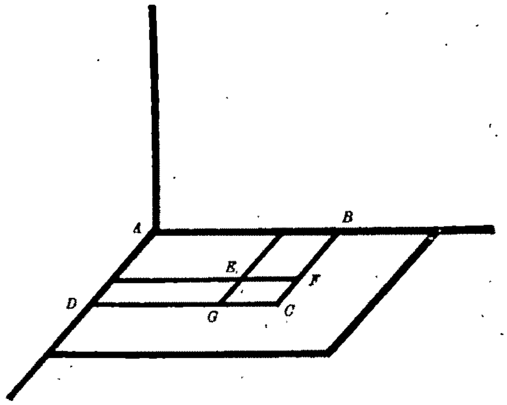

FIG. 14.

and if for every point (x, y) of this region two corresponding difference quotients are always equal, that is if

$$\frac{F(x+h, y+k) - F(x, y+k) - F(x+h, y) + F(x, y)}{hk} =$$

$$\frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk},$$

these values being finite, then $F(x, y)$ and $f(x, y)$ can differ at most by a constant quantity.

In order to prove this, we consider the function

$$\psi(x, y) = c(x-\alpha)(y-a) + F(x, y) - f(x, y) - [F(\alpha, a) - f(\alpha, a)]. \quad (1)$$

Let $DF(x, y)$, $Df(x, y)$, and $D\psi(x, y)$, be the corresponding difference quotients and let c be any positive quantity. Then $\psi(x, y)$ can never be negative in the region (A). In order to prove this we assume the contrary and suppose that $\psi(x, y)$ can be negative at some point $(\bar{x}, \bar{y})$ of the given region. Let this be the point C and consider now the rectangle $A(\alpha, a)$, $B(\bar{x}, a)$, $C(\bar{x}, \bar{y})$, $D(\alpha, \bar{y})$, see figure 14.

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* Cf. L. Scheeffer: Acta Mathematica, Vol. 5 (1884), pages 84 and 183.

Within this rectangle the points (x', y') at which (x, y) is not negative are such that the absolute value of $\sqrt{[x'^2 + y'^2]}$ has an upper limit which is finite. Suppose that this occurs at the point $(\bar{x}', \bar{y}') = E$. Since $\psi(x, y)$ is continuous at every point of region (A) in (x, y) together, it follows that $\psi(\bar{x}', \bar{y}') = 0$. But $\bar{x}' < \bar{x}$ and $\bar{y}' < \bar{y}$. Moreover we have for every point of the rectangle *EFCG*

$$\psi(x, y) - \psi(\bar{x}', \bar{y}') = 0.$$

From this follows that $D\psi(\bar{x}', \bar{y}') = 0$. On the other hand, $Df(x, y)$ and $D\psi(x, y)$ are both finite, and hence we have from (1) that $D\psi(x, y) = c$ for every point (x, y) of region (A). This last statement is surely true for

$$D\psi(x, y) = Dc(x - \alpha)(y - \alpha) + DF(x, y) - Df(x, y) = Dc(x - \alpha)(y - \alpha).$$

But $Dc(x - \alpha)(y - \alpha) = c$. We are thus led to a contradiction, and since this holds for every positive value of c, it follows that (x, y) can never be negative. Hence

$$F(x, y) - f(x, y) - [F(\alpha, \alpha) - f(\alpha, \alpha)]$$

cannot ever be negative. In a like manner we can prove that this difference cannot ever be positive and hence it must be zero. This proves the lemma.

This lemma may be extended as follows:

Lemma 2.—If the conditions of lemma 1 hold, with the exception that the two difference quotients are not equal to each other at an enumerable set of points, that is, are not everywhere dense in any sub-region of region (A), then the two given functions can differ at most by a constant quantity.

Let L be a set of exceptional points, and suppose that L is not everywhere dense in any sub-region of region (A). We then have a set D of sub-regions d which do not contain any of the exceptional points, and we can thus determine a closed set of points C, with respect to which L is everywhere dense. In every sub-region d the preceding theorem holds, so that for every sub-region d' contained wholly within d the difference $F(x, y) - f(x, y)$ is constant. But this difference is continuous in (x, y) together and hence it follows that it is constant also in the sub-region d. If this is so, then $F(x, y) - f(x, y)$ is either a constant or it is "streckenweise" discontinuous. (See Schoenfliess: Punktmannigfaltigkeiten, p. 166). If L is enumerable, then the above difference is constant, while if L has the "Mächtigkeit" of the continuum then this difference is "streckenweise" discontinuous. This proves the lemma.

We are now ready to take up our final theorem.

Theorem 3.—If we have

$$S_n(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + \ldots + u_n(x, y),$$

where $\lim\limits_{n=\infty} S_n(x, y) = S(x, y)$ for every (x, y) of

$$\alpha \leqq x \leqq \beta, \quad a \leqq y \leqq b, \tag{A}$$

and if $S(x, y)$ is completely defined and is always less in absolute value than some finite quantity, and if both the simultaneous integrals $\int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy$ and $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ exist, then the sufficient condition that

$$\lim_{n=\infty} \int_\alpha^\beta \int_a^b S_n(x, y)\, dxdy = \int_\alpha^\beta \int_a^b S(x, y)\, dxdy$$

where $\int_\alpha^\beta \int_a^b S(x, y)\, dxdy$ is finite, is that the set of points (x', y') for which we cannot assign a circle of radius $r > 0$ and an integer m, such that for every point (x, y) of this circle and for every $n > m$,

$$|S_n(x, y)| < G \qquad (G \text{ finite})$$

is enumerable.

In this case we have in the plane $1/n = z$ certain points (x', y') in whose neighborhood $S_n(x, y)$ does not have a finite upper limit. About such a point as center we can therefore construct a circle of radius $r > 0$, and at the same time find a plane $z = 1/m_s$, such that for every point (x, y) of this circle and for every value of $p = 1, 2, 3, \ldots$,

$$|S_{m_s+p}(x, y)| > G. \qquad (G \text{ finite})$$

This set of points (x', y') must evidently be a closed set, for it is a set for which the limit of the oscillation k is greater than some fixed finite number. We have now by hypothesis

$$\int_a^x \int_a^y S(x, y)\, dxdy = \theta(x, y)$$

and

$$\lim_{n=\infty} \int_a^x \int_a^y S_n(x, y)\, dxdy = \phi_0(x, y).$$

Now let (x, y) be some point not belonging to the set (x', y'). There exists then about it a circle of radius $r > 0$ for which

$$|S_{m_s+p}(x, y)| < G.$$

Consider now the ratios

$$\frac{\theta(x+h, y+k)-\theta(x+h, y)-\theta(x, y+k)+\theta(x, y)}{hk}$$

and

$$\frac{\phi_0(x+h, y+k)-\phi_0(x+h, y)-\phi_0(x, y+k)+\phi_0(x, y)}{hk}$$

where $(x+h,\ y+k)$ is to be *a* point of the above circle. Choose an arbitrarily small $\lambda>0$; we can then find a plane $z=1/m_s$, such that for every value of $p=1, 2, 3, \ldots$

$$\frac{\phi_0(x+h, y+k)-\phi_0(x+h, y)-\phi_0(x, y+k)+\phi_0(x, y)}{hk}=$$

$$\frac{\phi_{m_s+p}(x+h, y+k)-\phi_{m_s+p}(x+h, y)-\phi_{m_s+p}(x, y+k)+\phi_{m_s+p}(x, y)}{hk}+$$

$$-1<\mu<1,$$

since $\lim\limits_{n=\infty} \phi_{m_s+p}(x, y)=\phi_0(x, y)$ for every point $(x,\ y)$. But we have by hypothesis

$$\frac{\phi_{m_s+p}(x+h, y+k)-\phi_{m_s+p}(x+h, y)-\phi_{m_s+p}(x, y+k)+\phi_{m_s+p}(x, y)}{hk}=$$

$$1/hk\int_x^{x+h}\int_y^{y+k} S(x, y)\, dxdy+\mu\lambda$$

$$-1<\mu<1,$$

where λ is independent of $x,\ y,\ h$, and k. Since now $|S_n(x, y)|<G$ at the points in question, it follows from theorem 2 that

$$1/hk\int_x^{x+h}\int_y^{y+k} S_{m_s+p}(x, y)\, dxdy=1/hk\int_x^{x+h}\int_y^{y+k} S(x, y)\, dxdy+\mu'\frac{\lambda}{h}$$

$$-1<\mu'<1.$$

Hence

$$\frac{\phi_0(x+h, y+k)-\phi_0(x+h, y)-\phi_0(x, y+k)+\phi_0(x, y)}{hk}=$$

$$1/hk\int_x^{x+h}\int_y^{y+k} S(x, y)\, dxdy+\mu'\frac{\lambda}{h}+\mu\lambda$$

$$=\frac{\theta(x+h, y+k)-\theta(x+h, y)-\theta(x, y+k)+\theta(x, y)}{hk}+\lambda\left\{\mu+\frac{\mu'}{h}\right\}.$$

which holds for every x, $x+h$, y, and $y+k$, determined as above. We thus obtain two different quotients which are equal to each other at all points of region (A), excepting at a closed enumerable set of points, and hence by lemma 2 they can differ from each other at most by a constant quantity. Hence

$$\phi_0(x, y) = \theta(x, y) + \text{constant}.$$

But

$$\phi_0(x, y) = \theta(a, y) = 0$$

hence

$$\phi_0(x, y) = \theta(x, y),$$

which was to be proved.

By a proof similar to that just employed, we can show that the theorem holds also when we replace the function $S_n(x, y)$ by the general function $f(x, y, z)$.

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An Invariant Condition for Certain Automorphic Algebraic Forms.

By A. B. Coble.

§1. *Introduction.*

Prof. Moore[*] has shown that every finite collineation group in n variables has for an invariant a certain Hermitian form which may by a suitable transformation of the variables be taken as

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots . + x_n\bar{x}_n \tag{1}$$

Recalling that the collineations in contragredient variables are obtained by assuming the invariant

$$x_1u_1 + x_2u_2 + \ldots . + x_nu_n$$

and comparing this with (1) we see that when the group has been taken in Moore's canonical form, and the matrices of the collineation have a determinant value unity, then any matrix has for contragredient matrix its own conjugate. Or as Maschke[†] has pointed out, the minor of an element of the matrix is the conjugate of the element, and the complementary minor of a minor is the conjugate of the minor.

If the group has, for automorphic form of lowest degree, a certain unique algebraic form of order m in point coordinates, say f^m, the contragredient group has for automorphic form of lowest degree in contragredient variables a unique algebraic form of class m, say ϕ^m. If the group is in Moore's canonical form the coefficients of ϕ^m are the conjugates of those of f^m.

The collineation group being a thing of definite projective character, the same is true of the forms f^m and ϕ^m and it should be possible to define these forms by invariant conditions upon the general form of order m or of class m. This has actually been done for the three important groups in the linary domain and for the ternary G_{168}. But as f^m and ϕ^m occur simultaneously it would also seem possible to determine them simultaneously by invariant conditions upon a pair of general forms of order m and class m. In the following such an invariant condition will be given and will be seen to define some well-known automorphic forms.

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[*] Math. Ann. Bd. 50, p. 213.

[†] Math. Ann. Bd. 51, p. 253.

§2. *The Invariant Conditions.*

An algebraic form, f^{2n}, of even order $2n$ in q coordinates x may be written symbolically as $f^{2n} = (ax)^{2n}$; and a form γ^n of class n in contragredient variables as $\gamma^n = (u\gamma)^n$. γ^n defines then with regard to f^{2n} its "polar," c^n, where

$$c^n = (a\gamma)^n \, (ax)^n \equiv (cx)^n$$

Taking c^n and γ^n as contragredient "elements with the incidence condition, $(c\gamma)^n = 0$" we see that f^{2n} defines a *correlation* in these elements. The correlation is better exhibited by writing f^{2n} in polarized form

$$f^{n,n} \equiv (ax)^n \, (ay)^n \tag{2}$$

Since $f^{n,n}$ is symmetric in x and y, the correlation is of the type known as the *polarity*. It is not, however, the most general polarity. The latter would be given by the symmetrical form

$$L^{n,n} = (lx)^n \, (my)^n = (ly)^n \, (mx)^n \tag{3}$$

which differs from $f^{n,n}$ in not being the polarized form of a $2n$-*ic*.

The inverse polarity of (3) is also a polarity

$$\Lambda^{n,n} = (u\lambda)^n \, (v\mu)^n = (u\mu)^n \, (v\lambda)^n \tag{4}$$

of the general type, *i. e.*, not the polar of a form of class $2n$. But this is also true in general of the polarity (2), *i. e.*, the correspondence of γ^n to c^n given by (2) is reversed by a form $\Lambda^{n,n}$ of the *general* type. Only when the form of f^{2n} satisfies certain conditions will the inverse polarity $\Lambda^{n,n}$ be the polarized form of a quantic of class $2n$. And there is suggested the obvious query:

What is the nature of the form f^{2n} when the correspondence it establishes between forms γ^n and c^n is reversed by a form, $\phi^{2n} = (u\alpha)^{2n}$, i. e., if $(a\gamma)^n \, (ax)^n = (cx)^n$ then $(c\alpha)^n \, (u\alpha)^n \equiv \lambda \, (u\gamma)^n$ or if

$$(ax)^n \, (a\alpha)^n \, (u\alpha)^n \equiv \lambda \, (ux)^n \, (a\alpha)^{2n} \text{ ?}^{*} \tag{5}$$

We assume of course that the correspondence is not degenerate, *i. e.*, that the determinant of the nth derivatives of f^{2n} is not zero. This determinant, which is called in certain connections the catalecticant of f, will be denoted by A. The

---

[*] λ is here a numerical constant which by operation n times with the identical covariant (ux) we find to be $\dfrac{1}{n+q-1}$.

catalecticant bordered with two series of variables u and v in the combinations of class n gives $\Lambda^{n,n}$ and allowing the variables to coincide we obtain the form ϕ^{2n} as a contravariant of f^{2n}. From the symmetry of (5) we have that in the special case to be considered f^{2n} is the same contravariant of ϕ^{2n} that ϕ^{2n} is of f^{2n}. The forms f^{2n} and ϕ^{2n} will then be called a "covariant quantic-pair." To within a numerical factor A is the invariant $(a\alpha)^{2n}$.

The above query is answered completely in §3 for the binary domain and partially for the ternary domain in §4. Isolated results are further given and in §6 is found a treatment of a collineation group of order 25920 in five variables whose automorphic forms are obtained by the use of the invariant conditions just described.

§3. *The Forms f^{2n} and ϕ^{2n} in the Binary Domain.*

The distinction of contragredience is here trivial. We write both the forms f^{2n} and ϕ^{2n} [using as the symbol of a linear form $(a_1x_2 - a_2x_1)$] as

$$\begin{aligned} f^{2n} &\equiv (ax)^{2n} = (bx)^{2n} = \dots \\ \phi^{2n} &\equiv (\alpha x)^{2n} = (\beta x)^{2n} = \dots \qquad A \equiv (a\alpha)^{2n} \end{aligned} \tag{1}$$

and the condition (5) §2 becomes

$$(ax)^n\,(a\alpha)^n\,(\alpha y)^n \equiv \frac{1}{n+1}(a\alpha)^{2n}\,(xy)^n \tag{2}$$

From this we derive by the application of the Ω-process the further identities

$$\begin{aligned} &(a\alpha)^{n+1}\,(ax)^{n-1}\,(\alpha y)^{n-1} \equiv \frac{1}{n}\,(a\alpha)^{2n}\,(xy)^{n-1} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &(a\alpha)^{n+l}\,(ax)^{n-l}\,(\alpha y)^{n-l} \equiv \frac{1}{n-l+1}(a\alpha)^{2n}\,(xy)^{n-l} \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &l = 0,\ 1,\ \dots\ n \end{aligned} \tag{3}$$

We consider now the product of an invariant, $(ab)^{2n}$, of f^{2n} by the same invariant $(\alpha\beta)^{2n}$ of ϕ^{2n}

$$\begin{aligned} (ab)^{2n}\,.\,(\alpha\beta)^{2n} &\equiv [(a\alpha)\,(b\beta) - (a\beta)\,(b\alpha)]^{2n} \\ &= \sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}(a\alpha)^{2n-k}\,(b\beta)^{2n-k}\,(a\beta)^k\,(b\alpha)^k \\ &\quad + \sum_{k=n}^{k=2n}(-1)^k\binom{2n}{k}(a\beta)^k\,(b\alpha)^k\,(a\alpha)^{2n-k}\,(b\beta)^{2n-k} \end{aligned}$$

In every term of the sum occurs a term with $(a\alpha)$ or $(a\beta)$ to a degree n or greater and applying the identities (3) we have

$$
\begin{aligned}
(ab)^{2n} \,.\, (\alpha\beta)^{2n} = & \sum_{k=0}^{k=n-1} (-1)^k \binom{2n}{k} \frac{1}{k+1} (a\alpha)^{2n} \,.\, (b\beta)^{2n} \\
& + \sum_{k=n}^{k=2n} (-1)^k \binom{2n}{k} \frac{1}{2n-k+1} (a\beta)^{2n} \,.\, (b\alpha)^{2n}
\end{aligned}
\tag{4}
$$

All the symbols on the rights are symbols of A and we have

$$(ab)^{2n} \,.\, (\alpha\beta)^{2n} = \mu_n \,.\, A^2 \tag{5}$$

where μ_n is a numerical value

$$\mu_n = \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \frac{1}{k+1} + \sum_{k=n}^{2n} (-1)^k \binom{2n}{k} \frac{1}{2n-k+1} \tag{6}$$

We may obtain the value of the sum μ_n as follows:

$$
\begin{aligned}
\mu_n = & \; 2\left[1 - \tfrac{1}{2}\binom{2n}{1} + \tfrac{1}{3}\binom{2n}{2} \cdots + (-1)^{n-1}\frac{1}{n}\binom{2n}{n-1}\right] \\
& + (-1)^n \frac{1}{n+1}\binom{2a}{n}
\end{aligned}
$$

$$
\begin{aligned}
(2n+1)\,\mu_n = & \; 2\left[\binom{2n+1}{1} - \binom{2n+1}{2} \cdots + (-1)^{n-1}\binom{2n+1}{n} \right. \\
& \left. + (-1)^n \binom{2n+1}{n+1}\right] - (-1)^n \binom{2n+1}{n+1}
\end{aligned}
$$

$$
\begin{aligned}
2 - (2n+1)\,\mu_n = & \; 2\left[1 - \binom{2n+1}{1} + \binom{2n+1}{2} \cdots + (-1)^{n+1}\binom{2n+1}{n+1}\right] \\
& + (-1)^n \frac{2n+1}{n+1} \\
= & \; 2\left[(-1^{n-1} \frac{(2n)!}{(n-1)!\,(n+1)!}\right] + (-1)^n \left(\frac{2n+1}{n+1}\right) \\
= & \; (-1)^{n-1} \frac{(2n)!}{(n-1)!\,(n+1)!}\left[2 - \frac{2n+1}{n}\right] \\
= & \; (-1)^n \frac{(2n)!}{n!\,(n+1)!}
\end{aligned}
$$

$$\therefore \mu_n = \frac{2}{2n+1} + (-1)^{n+1} \frac{1}{(n+1)(2n+1)} \binom{2n}{n} \tag{7}$$

We see then that when $n = 1$ or $n = 3$, $\mu_n = 1$; when $n = 2$, $\mu_n = 0$; when $n > 3$, μ_n is greater in absolute value than 1.

By a process exactly similar to that by which we obtained (5) we prove that

$$\begin{aligned}(\alpha\beta)^{2n} \,.\, (ax)^{2n} &= \mu_n \,.\, A \,.\, (\alpha x)^{2n} \quad \text{and} \\ (ab)^{2n} \,.\, (\alpha x)^{2n} &= \mu_n \,.\, A \,.\, (ax)^{2n}\end{aligned} \tag{8}$$

Multiplying these two we see that

$$(\alpha\beta)^{2n} \,.\, (ab)^{2n} = \mu_n^2 \,.\, A^2$$

and comparing this with (5) we have at once

$$\mu_n = 1. \tag{9}$$

This argument is valid as long as $(ab)^{2n} \neq 0$ and $(\alpha\beta)^{2n} \neq 0$. But neither invariant vanishes unless $\mu_n = 0$ since A in (5) is by hypothesis not zero. The case $n = 2$ alone requires a special investigation. But it is well known that if a binary quartic $(ax)^4$ has its invariant $(ab)^4$ equal to zero, it has a covariant, the Hessian, which forms with it a "covariant quantic pair."* The quartic is in this case the well known "equianharmonic quartic" which is automorphic under a tetrahedral group.

We consider now the two cases for which $\mu_n = 1$. If $n = 1$, f^{2n} is a quadratic form and of course ϕ^{2n} coincides with it as we see also from (8). This case $n = 1$ occurs in every domain.

If $n = 3$, the covariant quantic pair is shown by (8) to be a single quantic which is subject to the condition (2) hence from (2) and (3)

$$\begin{aligned}(ax)^3 \, (ab)^3 \, (by)^3 &= \tfrac{1}{4}(ab)^6 \, (xy)^3 \\ (ax)^2 \, (ab)^4 \, (by)^2 &= \tfrac{1}{3}(ab)^6 \, (xy)^2\end{aligned} \tag{10}$$

Putting $x = y$ in this last we have

$$(ab)^4 \, (ax)^2 \, (bx)^2 \equiv 0. \tag{11}$$

But (11) is the condition that the sextic $(ax)^6$ be that which is unaltered by an octahedral group.† And it is easily verified that the octahedral sextic satisfies the identity (2). Hence

(12) *In the binary domain there are only three types of covariant quantic-pairs; the quadratic taken with itself and automorphic under an infinite group; the equianharmonic quartic and its Hessian automorphic under the tetrahedral group; that sextic taken with itself which is automorphic under the octahedral group.*

---

* Gordan, Invarianten Theorie, p. 211.

† Gordan, Invarianten Theorie, p. 211.

The case of the automorphic quartic presents nothing new. There are, however, some features in the case of the sextic that are novel. The general sextic $f^6 \equiv (ax)^6$ has for its catalecticant a symmetric determinant of the fourth order which, bordered with a series of variables x and a series y, gives rise to a symmetric form (12) $(dx)^3 (ey)^3 \equiv (dy)^3 (ex)^3$. In order that f^6 be the special form we are considering, this must be the polarized form of a sextic. The necessary and sufficient condition for this is given by the Clebsch–Gordan expansion as the identical vanishing of the quadratic form (13) $(de)^2 (dx) (ex)$. Both these forms are of the third degree in the coefficients. Since the sextic has only one covariant quadratic of this degree we may say that

(14) *The necessary and sufficient conditions that a sextic be octahedral is the non-vanishing of the catalecticant A and the identical vanishing of the quadratic covariant of third degree in the coefficients.*

This criterion is superior to the one usually given (see (11)) in that the number of conditions shows the exact number of constants in the form.

Connected with the tetrahedral quartic is a set of three irrational covariant quadratics which play an important role. They are defined as the quadratics which are their own polars as to the quartic. The set may be generalized for the octahedral sextic by seeking for cubics whose polars as to the sextic are their own cubic covariants. We take the sextic in its canonical form

$$f^6 = x_1^5 x_2 - x_2^5 x_1 \qquad \text{and the cubic as}$$

$$P = a_0 x_1^3 + 3a_1 x_1^2 x_2 + 3a_2 x_1 x_2^2 + a_3 x_2^3 .$$

The polar of P as to f^6 is the cubic covariant of P if

$$\begin{aligned} a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 &= \lambda a_2 \\ 3(a_0 a_1 a_3 - 2a_0 a_2^2 + a_1^2 a_2) &= -\lambda a_3 \\ 3(a_0 a_2 a_3 - 2a_1^2 a_2 + a_1 a_3^2) &= \lambda a_0 \\ a_0 a_3^2 - 3a_1 a_2 a_3 + 2a_2^3 &= -\lambda a_1 . \end{aligned} \tag{15}$$

We may also add

$$a_0^2 a_3^2 + 4a_0 a_2^3 + 4a_1^3 a_3 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 = R \neq 0 .$$

This system of equations* may be replaced by the system

$$\begin{aligned} &1^\circ \; a_0 a_1 + a_2 a_3 = 0 \qquad && 4^\circ \; a_0 a_2^3 - a_1^3 a_3 = 0 \\ &2^\circ \; a_1^4 + a_2^4 = 0 && 5^\circ \; a_0^2 a_3^2 + 3a_1^2 a_2^2 - 4a_0 a_2^3 = 0 \\ &3^\circ \; a_0^4 + a_3^4 = 0 && 6^\circ \; (a_0 a_3 - a_1 a_2)^2 = \frac{R}{3} . \end{aligned} \tag{16}$$

---

* The following solution was supplied by Mr. Landry.

We see at once that a_4 cannot be zero unless P vanishes identically. Taking δ and ε so that $\delta^4 = -1$ and $\varepsilon^4 = -1$ we have from 2° and 3° that

$$a_2 = \varepsilon a_1 \qquad a_3 = \delta a_0.$$

Substituting these values in 1° or 4° we find

$$\delta\varepsilon = -1 \qquad \therefore \varepsilon = -\frac{1}{\delta} = \delta^3.$$

Substituting the values of a_2, a_3, and ε in 5° and putting $\frac{\delta a_1}{a_0} = \alpha$ the equation 5° becomes

$$3\alpha^4 + 4\alpha^3 + 1 = (\alpha + 1)^2 (3\alpha^2 - 2\alpha + 1) = 0$$

$\alpha = -1$ makes R vanish and must be discarded.

The other factor has roots $\alpha_1 = \frac{1 + i\sqrt{2}}{3}$ and $\alpha_2 = \frac{1 - i\sqrt{2}}{3}$. For the root α_2 and δ any one of the four primitive eighth roots of unity we find four cubics

$$\begin{aligned}
P_1 &= x_1^3 + 3\delta\alpha_2\, x_1^2x_2 + 3\delta^6\alpha_2\, x_1x_2^2 + \delta^7x_2^3 \\
P_2 &= x_1^3 + 3\delta^3\alpha_2\, x_1^2x_2 + 3\delta^2\alpha_2\, x_1x_2^2 + \delta^5x_2^3 \\
P_3 &= x_1^3 + 3\delta^5\alpha_2\, x_1^2x_2 + 3\delta^6\alpha_2\, x_1x_2^2 + \delta^3x_2^3 \\
P_4 &= x_1^3 + 3\delta^7\alpha_2\, x_1^2x_2 + 3\delta^2\alpha_2\, x_1x_2^2 + \delta x_2^3
\end{aligned}$$

and for the root α_1 we find four others

$$\begin{aligned}
Q_1 &= x_1^3 + 3\delta^5\alpha_1\, x_1^2x_2 + 3\delta^6\alpha_1\, x_1x_2^2 + \delta^3x_2^3 \\
Q_2 &= x_1^3 + 3\delta^7\alpha_1\, x_1^2x_2 + 3\delta^2\alpha_1\, x_1x_2^2 + \delta x_2^3 \\
Q_3 &= x_1^3 + 3\delta\alpha_1\, x_1^2x_2 + 3\delta^6\alpha_1\, x_1x_2^2 + \delta^7x_2^3 \\
Q_4 &= x_1^3 + 3\delta^3\alpha_1\, x_1^2x_2 + 3\delta^2\alpha_1\, x_1x_2^2 + \delta^5x_2^3.
\end{aligned}$$

Of this system we note that P_i and Q_i are cubic and cubic covariant. Each is the polar of the other as to f^6. But further the bilinear invariant of Q_i and P_k for $i \neq k$ is zero. This remarkable property is sufficient to define the system.

The two products $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ are each equal to

$$x_1^{12} - 33\, x_1^8x_2^4 - 33\, x_1^4x_2^8 + x_2^{12}$$

from which we see that when the sextic is taken as a regular octahedron* each set of four cubics represents the twelve middle points of the edges. These twelve points are arranged as follows: The octahedron has 4 planes of symmetry parallel to the 4 pairs of opposite faces. On each plane lie six mid-edge points.

---

* Klein—Ikosaeder—I, 2.

Three of these that are equidistant say 1, 2, 3 may be chosen as P_1, the other three say 4, 5, 6 as Q_1. Each of the other planes contains a point of 1, 2, 3 and a point of 4, 5, 6. We choose as P_2, P_3, P_4 those sets of equidistant mid-edge points in each plane which do not contain 1, 2, or 3. The other sets of three are Q_2, Q_3, Q_4 respectively.

The condition (14) also admits of an interpretation in three dimensians when the cubic space curve C^3 is taken as the bearer of the binary value system. A binary cubic represents a point or a plane through the point. A general sextic represents a quadric Q which, if taken as a point quadric, must be apolar to the web of quadrics circumscribed to C^3. The condition (14) requires that Q also be apolar to the net of quadrics inscribed to Q. We have then

If Q is a quadric apolar to the ∞^2 quadrics containing the planes of C^3, the necessary and sufficient condition that Q cut out on C^3 an octahedral set of six points is that Q also be apolar to the ∞^2 quadrics containing the points of C^3. The relation of Q and C^3 is perfectly dual. The planes common to C^3 and Q osculate C^3 at the points common to Q and C^3.

There exists a single pair of "Möbius' tetrahedra" P_i and Q_i such that each is a self-polar tetrahedron of Q. The joins of corresponding points of P_i and Q_i are chords of C^3 meeting C^3 in the 8 points that correspond to the cube. The planes of P_i and of Q_i cut out on C^3 the twelve points that correspond to the mid-edge points of the cube or octahedron.

We obtain convenient canonical forms for our covariant quantic pairs by assuming f and ϕ both to be unaltered by the permutations and changes of sign of the variables and then applying the definition of the quantic pair to determine the remaining constants. The three cases are then the following

$$
\begin{array}{lll}
n = 1 & f^2 = \phi^2 = x_1^2 + x_2^2 & \\
n = 2 & f^4 = x_1^4 + x_2^4 \pm 2i\sqrt{3}\,x_1^2x_2^2 & \phi^4 = x_1^4 + x_2^4 \mp 2i\sqrt{3}\,x_1^2x_2^2 \\
n = 3 & f^6 = \phi^6 = x_1^6 + x_2^6 - 5\,(x_1^4x_2^2 + x_1^2 \times x_2^4). &
\end{array}
$$

The latter is the octahedral sextic referred to a pair of opposite mid-edge points.

§4. *The Forms f^{2n} and ϕ^{2n} in the Ternary Domain.*

The simple arithmetical criterion [(9) §3] for a covariant quantic pair that was obtained for the binary domain does not admit of an immediate extension to the ternary domain. We shall treat the values of n separately and begin with n equal to 2.

In a former paper by the author* a certain class of biquadratic forms were treated which were defined in the same way as the above covariant form-pairs. The automorphic character of the resulting quartics was not recognized, however. We shall show first of all that the pair of quartics, f^4 and ϕ^4, are automorphic under the ternary group of collineations G_{168}. The method used is, in some respects, novel. The symbolic notation is that of Study's Ternären Formen.

The ternary quartic form, $f^4=(ax)^4$, after being polarized into $(ax)^2(ay)^2$ may be viewed as a degenerate symmetric quadri-quadric form. As such it is distinguished from the general symmetric form $(ax)^2(by)^2\equiv(ay)^2(bx)^2$ by the identical vanishing of its quadratic contravariant $(abu)^2$. If the ternary quadratic form be taken as fundamental space element, the biquadratic form appears as a quadric and under the general projective group, G_{35}, of this space has but three comitants which involve only one series of point or of dual coordinates. These are the quadric itself in point coordinates, the quadric in dual coordinates, and the discriminant of the quadric. In connection, then, with the given quartic $f^4=(ax)^4$ we have the invariant

$$A\equiv|a_{ik,jl}|\qquad ij,\text{ and }kl=11,\ 22,\ 33,\ 23,\ 31,\ 12 \tag{1}$$

and the form $(uA)^2(vB)^2\equiv(uB)^2(vA)^2$ which is defined by the identity

$$(ax)^2(aA)^2(uB)^2\equiv A\,.\,(ux)^2 \tag{2}$$

the three forms constituting the complete systen of $(ax)^2(ay)^2$ under G_{35}. They are of course also invariant under the projective group of the plane, G_8, a subgroup of G_{35}.

In A the element $a_{ij,kl}$ is distinguished by its position in the determinant from the element $a_{ik,jl}$. It seems useful in certain connections to preserve this distinction which will sometimes be emphasized by writing for the double suffices 11, 22, 33, 23, 31, 12 the single suffices 1, 2, 3, 4, 5, 6, respectively, in which case A appears as

$$|a_{i,k}|\qquad i,\text{ and }k=1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

Thus the latter notation makes it possible to write $(uA)^2(vB)^2$ as the result of an evectant operator upon A namely

$$(uA)^2(vB)^2\equiv(u\Omega^{(1)})^2(v\Omega^{(2)})^2 \tag{3}$$

---

[*] Trans. Amer. Math. Soc. Vol. 4, No. 1, pp. 65-85; 1903. This paper will be referred to hereafter as "Trans."

where $\Omega^{(1)}_{ij}\ \Omega^{(2)}_{kl} = \mu \dfrac{\partial}{\partial a_{r,s}} A$, the numerical coefficient μ and the suffices r and s being determined by ij and kl. The symmetric form $(uA)^2 (vB)^2$ determines the contravariant quartic form

$$\phi^4 \equiv (u\alpha)^4 \equiv (uA)^2 (uB)^2 \tag{4}$$

and the covariant quadratic form

$$C \equiv (Cx)^2 \equiv (ABx)^2 \tag{5}$$

On the other hand, both (4) and (5) are required to determine (3), as is evident from the formula

$$(uB)^2 (uB)^2 \equiv (u\alpha)^2 (v\alpha)^2 + \tfrac{1}{3}(Cuv)^2 \tag{6}$$

It is also evident from (6) that the identical vanishing of C is necessary and sufficient in order that (3) be the polarized form of a quartic (4). Hence

(7) *The necessary and sufficient condition that the quartics f^4 and ϕ^4 be a covariant quantic pair is the identical vanishing of the conic C of the fifth degree in the coefficient of f^4.*

From (3) C may also be written as the result of a differential operation upon A

$$\begin{aligned}(Cx)^2 = \Big[&\Big(\frac{\partial}{\partial a_{23}} - \frac{1}{2}\frac{\partial}{\partial a_{44}}\Big) x_1^2 + \Big(\frac{\partial}{\partial a_{31}} - \frac{1}{2}\frac{\partial}{\partial a_{55}}\Big) x_2^2 + \Big(\frac{\partial}{\partial a_{12}} - \frac{1}{2}\frac{\partial}{\partial a_{66}}\Big) x_3^2 \\ &+ \Big(\frac{1}{2}\frac{\partial}{\partial a_{56}} - \frac{\partial}{\partial a_{14}}\Big) x_2 x_3 + \Big(\frac{1}{2}\frac{\partial}{\partial a_{64}} - \frac{\partial}{\partial a_{25}}\Big) x_3 x_1 \\ &+ \Big(\frac{1}{2}\frac{\partial}{\partial a_{45}} - \frac{\partial}{\partial a_{36}}\Big) x_1 x_2 \Big] A\end{aligned} \tag{8}$$

Writing this differential operator symbolically also we have

$$(Cx)^2 = (\omega x^2) \tag{9}$$

where ω_{ik} is then the symbol for an operation which is understood to be carried out on A.

We consider now the pencil of symmetric quadri-quadric forms

$$\lambda\, (ay)^2\, (az)^2 + \mu\, (\gamma y z)^2 \tag{10}$$

and form the invariant A of the pencil. By the use of the operator $(\omega x)^2$ defined in (8) this invariant takes the form

$$[\lambda^6 + (\omega\gamma)^2 \lambda^5 \mu + \frac{1}{2!} (\omega\gamma)^2 (\omega'\gamma')^2 \lambda^4\mu^2 + \frac{1}{3!} (\omega\gamma)^2 (\omega'\gamma')^2 (\omega''\gamma'')^2 \gamma^3 \mu^3 + \\ \cdots\cdots + \frac{1}{5!} (\omega\gamma)^2 (\omega'\gamma')^2 \cdots (\omega^{\mathrm{v}}\gamma^{\mathrm{v}})^2 \mu^6] A \tag{11}$$

This repetition of the operation ω is permissible since in the minors A, as in A itself, the distinction between such coefficients as $a_{44} = a_{23\,23}$ and $a_{23} = a_{22\,33}$ is preserved. The coefficient of each power of λ in (11) is a simultaneous invariant of $(ax)^4$ and $(u\gamma)^2$. Equating (9) to zero, we obtain in general six values of $\frac{\lambda}{\mu}$ to each of which corresponds a conic $(u\delta)^2$ such that

$$\lambda (a\delta)^2 (ay)^2 \equiv -\mu (\gamma\delta y)^2$$

i. e., the polar conic of $(u\delta)^2$ as to f^4 is its intermediate* with the fixed conic $(u\gamma)^2$. The particular case for which the conic $(u\gamma)^2$ is the square of a point x is of interest. There is then a web of conics $(u\delta)^2$—all pairs of points containing x—such that $(x\delta y)^2 \equiv 0$. Three values of λ in (11) are zero and the equation reduces to

$$\lambda^3 A + (\omega x)^2 \lambda^2 + \frac{1}{2} (\omega x)^2 (\omega' x)^2 \lambda\mu^2 + \frac{1}{6} (\omega x)^2 (\omega' x)^2 (\omega'' x)^2 \mu^3 = 0. \tag{12}$$

The coefficients of λ in (12) are important covariants of the quartic. Performing the operations indicated by the symbols ω we have

$$\begin{aligned} (\omega x)^2 &= (Cx)^2 = \left(2A_{\substack{2\\3}} - \frac{1}{2} A_{\substack{4\\4}}\right) x_1^2 + \cdots \\ (\omega x)^2 (\omega' x)^2 &= \left(A_{\substack{23\\32}} - A_{\substack{42\\43}}\right) x_1^4 + \cdots \\ (\omega x)^2 (\omega' x)^2 (\omega'' x)^2 &= -1/2\, A_{\substack{234\\324}} x_1^6 + \cdots \end{aligned} \tag{13}$$

the notation for the minors of A being that of Gordan† (Kerschensteiner). By means of these leading coefficients the covariants may be indentified with the following symbolic forms

---

* "Trans." p. 70.

† Invarianten theorie, Erster Theil, p. 47.

$$\begin{aligned} C &= (\omega x)^2 &&= -\frac{1}{32}(abc)^2(ade)^2(dbc)(ebc)(dx)(ex) \\ S &= (\omega x)^2(\omega' x)^2 &&= -\frac{1}{12}(abc)(abd)(acd)(bcd)(ax)(bx)(cx)(dx) \quad (14) \\ H &= (\omega x)^2(\omega' x)^2(\omega'' x)^2 &&= \frac{1}{2}(abc)^2(ax)^2(bx)^2(cx)^2 \end{aligned}$$

We seek first to interpret the vanishing of the discriminant of the equation (12) *i. e.* of

$$2(3AS-2C^2)(2HC-S^2)-(3AH-SC)^2=0 \qquad (15)$$

For a given x to each root of (12) corresponds a conic $(u\delta)^2$ whose polar as to f^4 is the two tangent of $(u\delta)^2$ through x. The point equation of $(u\delta)^2$ is the conic C formed for the pencil

$$\lambda^2(\omega y)^2+\lambda\mu(\omega x)^2(\omega' y)^2+\frac{1}{2}\mu^2(\omega x)^2(\omega' x)^2(\omega'' y)^2+\psi(x, y)=0 \qquad (16)$$

where $\psi(x, y)$ vanishes identically for $x=y$. Hence, if (16) passes through x, the first three terms separately vanish for $x=y$. But these terms are the derivative of 12 as to μ, *i. e.*, (12) has a double root and (15) is satisfied by x. Suppose now u is a line of the quartic ϕ^4 touching ϕ^4 at x. Then* u touches its associate* conic $(u\delta)^2$ at the point x and for a proper λ and μ $(u\delta)^2$ satisfies the identity

$$\lambda(a\delta)^2(ay)^2\equiv\rho(uy)^2\equiv-\mu(x\delta y)^2$$

and also passes through x. Hence x is a point of (15) and we have

(17) *The equation* (15) *is that of the contravariant quartic* $\phi^4=(u\alpha)^4$ *in point coordinates.* We can easily show further that

(18) *The three pairs of lines through a general point* x *such that each pair touches its associate conic form the sextic covariant of the binary quartic formed by the four tangents from* x *to* ϕ^4.

If x is taken at a cusp of $(u\alpha)^4$, the four tangents include the cusp tangent three times, the sextic covariant the cusp tangent six times and (11) has a triple root. Hence

---

* "Trans." p. 67, footnote.

(19) *The cusps of* ϕ^4 *are the intersections of* $3AS - 2C^2 = 0$ *and* $3(AH - SC) = 0$.

Returning to the general invariant (9) we can express in terms of its symbols other comitants of f^4; for example

$$
\begin{array}{llll}
(\omega\omega' u)^2 & \equiv C_{402} & (\omega\omega' u)^2 (\omega x)^2 \equiv C_{322} & \\
(\omega\omega'\omega'')^2 & \equiv C_{300} & (\omega\omega' u)^2 (\omega''\omega''' u)^2 \equiv C_{204} & \qquad (20) \\
(\omega\omega' u)^2 (\omega'' x)^2 (\omega''' x)^2 & \equiv C_{242} & (\omega\omega'\omega'')^2 (\omega''' x)^2 (\omega^{iv} x)^2 \equiv C_{140} &
\end{array}
$$

C^{ikl} denoting a comitant of f^4 of degree i, order k, and class l. All of the comitants enumerated in (14) and (20) are expressible in terms of A, and the coefficients of $\phi^4 = (u\alpha)^4$ and of $(Cx)^2$ as follows:—

If λ and μ are values which satisfy (11) a conic $(u\delta)^2$ exists which satisfies the identity

$$(\gamma\delta y)^2 \equiv -\frac{\lambda_1}{\mu_1} (a\delta)^2 (ay)^2$$

Operating with both sides of this identity on $(uA)^2 (vB)^2$ and making use of (2) there results a new identity

$$(A\gamma\delta)^2 (uB)^2 \equiv -\frac{\lambda_1}{\mu_1} \cdot A \cdot (u\delta)^2$$

$-\dfrac{\lambda_1}{\mu_1} A$ is then a root of the characteristic equation of the connex (2, 2)

$$(A\gamma x)^2 (uB)^2$$

which can be written as*

$$\rho^6 - \mathrm{I}_1 \rho^5 + \mathrm{I}_2 \rho^4 - \mathrm{I}_3 \rho^3 + \mathrm{I}_4 \rho^2 - \mathrm{I}_5 \rho + \mathrm{I}_6 = 0 \text{ if}$$

$$\mathrm{I}_k \equiv \frac{1}{k!} \begin{vmatrix} (AB\gamma)^2 & (AB'\gamma)^2 & (AB''\gamma)^2 & \dots \\ (A'B\gamma)^2 & (A'B'\gamma)^2 & (A'B''\gamma)^2 & \dots \\ (A''B\gamma)^2 & (A''B'\gamma)^2 & (A''B''\gamma)^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \qquad (21)$$

the determinant having k rows and columns. Putting $\rho = -\dfrac{\lambda_1}{\mu_1} A$ in the characteristic equation and dividing out by A^5 it becomes

$$A\lambda_1^6 + I_1 \lambda_1^5 \mu_1 + \frac{I_2}{A} \lambda_1^4 \mu_1^2 + \frac{I_3}{A^2} \lambda_1^3 \mu_1^3 + \dots + \frac{I_6}{A^5} \mu_1^6 = 0$$

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* "Trans." p. 73-4.

λ_1, and μ_1, satisfy (22) due to their satisfying (11) whence (11) and (22) are the same equation. Equating coefficients and introducing the values of I_k from (21) we have the relations

$$\begin{aligned}(\omega\gamma)^2 &= (AB\gamma)^2\\ A\,.\,(\omega\gamma)^2(\omega'\gamma')^2 &= [(AB\gamma)^2]^2-(AB'\gamma)^2(A'B\gamma')^2\\ A^2\,.\,(\omega\gamma)^2(\omega'\gamma')^2(\omega''\gamma'')^2 &= [(AA\gamma)^2]^3-3(AB\gamma)^2\,.\,(AB'\gamma)^2(A'B\gamma')^2\\ &\quad+2(AB'\gamma)^2(A'B''\gamma')^2(A''B\gamma'')^2.\end{aligned}\tag{23}$$

. . . .

The second of these identities will be treated first. It is of the second degree in the coefficients of $(u\gamma)^2$ but may be polarized by the introduction of a new set of coefficients of $(u\delta)^2$. The symbols γ and δ may then be replaced by x and y and we have

$$A\,.\,(\omega x)^2(\omega' y)^2\equiv(Cx)^2\,.\,(Cy)^2-(AB'x)^2(A'By)^2.\tag{24}$$

From the identity (6) we have

$$(uA)^2(vB)^2\,.\,(u'A')^2(v'B')^2\equiv\left[(u\alpha)^2(v\alpha)^2+\frac{1}{3}(Cuv)^2\right]$$
$$[(u'\alpha')^2(v'\alpha')_2+1/3\,(C'u'v')^2.$$

Adjoining on both sides the symbols u and v' into symbols x and then the symbols v and u' into symbols y there results on the left $(AB'x)^2(A'By)^2$ and on the right

$$(\alpha\alpha'x)^2(\alpha\alpha'y)^2+2/3\,(\alpha xy)^2(C\alpha)^2+\frac{1}{9}[-2(C'C'xy+3(C'x)^2.(C'y)^2].\tag{25}$$

Substituting this value in (24) we have the final identity

$$A\,.\,(\omega x)^2(\omega y)^2\equiv-(\alpha\alpha'x)^2(\alpha\alpha'y)^2-2/3\,(\alpha xy)(C\alpha)^2+\frac{1}{9}[2(CC'xy)^2$$
$$+6(Cx)^2(C'y)^2].\tag{26}$$

This gives at once the two special identities

$$\begin{aligned}A\,.\,(\omega x)^2(\omega' x)^2 &= \frac{2}{3}[(Cx)^2]^2-(\alpha\alpha'x)^4\\ A\,.\,(\omega\omega' u)^2 &= 4/3\,(CC'u)^2-2(u\alpha)^2(C'\alpha)^2.\end{aligned}\tag{27}$$

A similar process may be carried through for the third identity. The values obtained are rather cumbersome and we give only the following special results

$$A^2\,.\,(\omega x)^2(\omega' y)^2(\omega'' z)^2\equiv 2(\alpha\alpha'x)^2(\alpha'\alpha''y)^2\,\alpha''\alpha z)^2+\chi(C).\tag{28}$$

where $X(C)$ indicates a sum of comitants of at least the first degree in the coefficients of $(Cx)^2$

$$A^2 \,.\, (\omega x)^2 (\omega' x)^2 (\omega'' x)^2 \equiv 2\,(\alpha\alpha' x)^2 (\alpha'\alpha'' x)^2 (\alpha''\alpha x)_2 \\ - (Cx)^2 \,.\, (\alpha\alpha' x)^4 + 2/9\,[(Cx)^2]^2. \tag{29}$$

From (29) and (27) the point equation of $(u\alpha)^4$ can be calculated from the form $[(\alpha\alpha' x)^4]^3 - 6[(\alpha\alpha' x)^2 (\alpha'\alpha'' x)^2 (\alpha''\alpha x)^2]^2$ and is found to coincide with that of (15).

Adjoining in (28) the variables of y and z to the variable u we have

$$A^2 \,.\, (\omega x)^2 (\omega'\omega'' u)^2 \equiv 2(\alpha\alpha'\alpha'') (\alpha\alpha' x)^2 (u\alpha'')^2 + \chi_1(C). \tag{30}$$

A simple calculation of the value of $(\omega x)^2 (\omega'\omega'' u)^2$ from the original definition of the symbols ω gives

$$(\omega x)^2 (\omega'\omega'' u)^2 = -\frac{1}{48}\,[(aa'a'')^4 (ux)^4 + 6(aa'a'')^2 (aa'u)^2 (a''x)^2]. \tag{31}$$

We suppose now $(Cx)^2$ identically zero.

The second formula of (27) shows that the contravariant conic of fourth degree, C_{402}, vanishes. This necessitates the vanishing of the entire pencil of covariant conics of fifth degree for C itself and the polar of C_{402} as to $(ax)^4$ are two distinct members of the pencil. *Hence, if $(Cx)^2 \equiv 0$, all the quadratic comitants of f^4 of fourth and fifth degrees also vanish identically.*

In (30), $\chi_1(C)$ vanishes and from (30) and (31) we have

$$-\frac{A^2}{48}\,[(aa'a'')^4 (ux)^2 + 6(aa'a)^2 (aa'u)^2 ({}''x)^2] = 2(\alpha\alpha'\alpha'') (\alpha\alpha' x)^2 (u\alpha'')^2. \tag{32}$$

Taking the bilinear invariant as to u and x on both sides we obtain

$$-A^2 \,.\, (aa'a'')^4 = 8(\alpha\alpha'\alpha''). \tag{33}$$

We recall also the theorem * that if the I_1 of $(u\alpha)^4$ is the I_2 of $(ax)^4$, the I_2 of $(u\alpha)^4$ is $\left(\frac{A^4}{64}\right)^n$ times the I_1 of $(ax)^4$ where n is the degree of I_1 in the coefficients. Treating the right hand member of (32) as I and remembering that the A of $(u\alpha)^4$ is now $\frac{A^5}{64}$ the theorem gives rise to the new identity

$$-\frac{1}{48}\left(\frac{A^5}{64}\right)^2 [(\alpha\alpha'\alpha'')^4 (ux)^2 + 6(\alpha\alpha'\alpha'')^2 (\alpha\alpha' x)^2\, u a^2] \\ = \left(\frac{A^4}{64}\right)^3 2(aa'a'')^2 (aa'u)^2 (a''x)^2. \tag{34}$$

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* "Trans." p. 68.

Replacing in this $(\alpha\alpha'\alpha'')^4$ by its value in (33) we find

$$-\frac{A^2}{48}\left[-2(aa'a'')^4\,(ux)^2+24(aa'a'')\,(a'a''u)\,(ax)^2\right]=2(\alpha\alpha'\alpha'')^2\,(\alpha'\alpha''x)^2\,(u\alpha)^3. \quad (35)$$

Comparing (35) and (32) we have finally

$$6(aa'a'')^3\,(a'a''u)^2\,(ax)^2\equiv(aa'a'')^4\,(ux)^2. \quad (36)$$

If then $C\equiv 0$ the identity (36) is true. Conversely, if (36) is true, $C\equiv 0$. For the quartic defined by the identity (36) whose complete system has been set forth by Gordan * has no quadratic comitants. The condition (36) may then be replaced by the condition, $C\equiv 0$. (36) is further shown by Gordan (loc. cit.) to be the necessary and sufficient condition that f^4 be automorphic under the finite ternary collineation group G_{168}. We have then

(37) *The vanishing of the covariant conic C of fifth degree is the necessary and sufficient condition that the quartics f^4 and ϕ^4 be automorphic under the group G_{168}.*

We obtain a convenient canonical form of the covariant quantic-pair by assuming quantics unaltered by a cyclical advance of the variables and by changes of sign of the variables. Taking f^4 as

$$a(x_1^4+x_2^4+x_3^4)+6b(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2)$$

and ϕ^4 as

$$\alpha(u_1^4+u_2^4+u_3^4)+6\beta(u_2^2u_3^2+u_3^2u_1^2+u_1^2u_2^2)$$

we find as the conditions that the polar of any conic as to f^4 shall have for polar as to ϕ^4 the given conic

$$a\alpha+2b\beta=4b\beta=\lambda\neq 0$$
$$a\beta+b\alpha+b\beta=0.$$

Putting $a=\alpha=1$ we find b and β as the two roots of

$$2\mu^2+\mu+1=0. \quad (38)$$

Putting then

$$\mu_1=\frac{-1+i\sqrt{7}}{4}\,\mu_2=\frac{-1-i\sqrt{7}}{4}.$$

we have for the covariant quantic-pair

$$\left.\begin{aligned}f^4&\equiv x_1^4+x_2^4+x_3^4+6\mu_1\,(x_2^2x_3^2+x_3^2x_1^2+x_1^2x_2^2)\\ \phi^4&\equiv u_1^4+u_2^4+u_3^4+6\mu_2\,(x_2^2u_3^2+u_3^2u_1^2+u_1^2u_2^2).\end{aligned}\right\} \quad (39)$$

This canonical form of the quartic unaltered by G_{168} has been mentioned by

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* Math. Ann., Vol. 17, p. 217 and p. 359.

Wiman[*] and the group itself has been treated in great detail by Klein and Fricke.[†] It may be obtained, however, with so little effort from the forms (39) that it seems worth while to carry through the work here.

We will seek first a correlation K which will transform f^4 into ϕ^4. The well known properties of the octahedral group G_{24} which evidently leaves f^4 and ϕ^4 unaltered suggests a trial of the correlation

$$K \equiv \begin{cases} x_1 = u_1 + u_2 \\ x_2 = u_1 - u_2 \\ x_3 = \qquad \rho u_3. \end{cases}$$

In order that K transform f^4 into ϕ^4, ρ must satisfy the equations

$$2 + 6\mu_1 = \rho^4,\ 1 - \mu_1 = \mu_1\rho^2,\ \mu_2 = \frac{2\mu_1}{\rho^2}$$

whence $\rho = \pm \dfrac{1}{\mu_2}$. Choosing the negative sign and forming the combination KTK^{-1} where T is the collineation $u_1 = u_3'$, $u_2 = u_2'$, $u_3 = -u_1'$, which leaves ϕ^4 unaltered, we have

$$KTK^{-1} \equiv x' = \begin{pmatrix} 1 & -1 & -2\mu_2 \\ -1 & 1 & -2\mu_2 \\ 2\mu_1 & 2\mu_1 & 0 \end{pmatrix} x'$$

a collineation which leaves f^4 unaltered. Introducing a factor $\frac{1}{2}$ to make the determinant unity, interchanging the last two rows of the matrix and changing the sign of the first we have a collineation S of period 7 which also leaves f unaltered S and its powers are

$$S \equiv x' = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & 2\mu_2 \\ 2\mu_1 & 2\mu_1 & 0 \\ -1 & 1 & -2\mu_2 \end{pmatrix} x \qquad S^2 \equiv x' = \tfrac{1}{2}\begin{pmatrix} -2\mu_2 & -1 & 1 \\ 2\mu_2 & -1 & 1 \\ 0 & 2\mu_1 & 2\mu_1 \end{pmatrix} x$$

$$S^3 \equiv x' = \tfrac{1}{2}\begin{pmatrix} 2\mu_2 & 1 & 1 \\ 0 & -2\mu_1 & 2\mu_1 \\ 2\mu_3 & -1 & -1 \end{pmatrix} x \qquad S^4 \equiv x' = \tfrac{1}{2}\begin{pmatrix} 2\mu_1 & 0 & 2\mu_1 \\ 1 & -2\mu_2 & -1 \\ 1 & 2\mu_2 & -1 \end{pmatrix} x$$

$$S^5 \equiv x' = \tfrac{1}{2}\begin{pmatrix} -2\mu_1 & 2\mu_1 & 0 \\ -1 & -1 & 2\mu_2 \\ 1 & 1 & 2\mu_2 \end{pmatrix} x \qquad S^6 \equiv x' = \tfrac{1}{2}\begin{pmatrix} -1 & 2\mu_2 & -1 \\ 1 & 2\mu_2 & 1 \\ 2\mu_1 & 0 & -2\mu_1 \end{pmatrix} x$$

$$S^7 = x' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$$

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[*] Math. Annalen, Vol. 47, 1896, p. 536.

[†] Klein-Fricke; Elliptischen Modulfunctionen.

The G_7 made up of the powers of S, combined with the G_{24} consisting of the permutations and changes of sign of the variables gives the entire G_{168} or more briefly $G_{168} = G_{24} \, . \, G_7$.

Any one of the 168 matrices is obtained from one of the above 7 by interchanging the rows or changing the signs of a row throughout. The factor $\frac{1}{2}$ before the matrix is understood as affecting each element and the group is then in Moore's canonical form.

In the theory of the group as developed by Klein-Fricke two sets of seven conics play an important role. We obtain the one set by transforming $x_1^2 + x_2^2 + x_3^2$ by G_7, the other by transforming this set by K. We find them to be

$$\begin{gathered} x_1^2 + x_2^2 + x_3^2 \\ -(\mu_2 + 1)\, x_1^2 - \frac{\mu_1}{2}(x_2^2 + x_3^2) \pm (2 + \mu_1)\, x_2 x_3 \\ -(\mu_2 + 1)\, x_2^2 - \frac{\mu_1}{2}(x_3^2 + x_1^2) \pm (2 + \mu_1)\, x_3 x_1 \\ -(\mu_2 + 1)\, x_3^2 - \frac{\mu_1}{2}(x_1^2 + x_2^2) \pm (2 + \mu_1)\, x_1 x_2 . \end{gathered} \tag{40}$$

The other set of seven is

$$\begin{gathered} 2\mu_1^2 u_1^2 + u_2^2 + u_3^2 \\ u_1^2 + 2\mu_1^2 u_2^2 + u_3^2 \\ u_1^2 + u_2^2 + 2\mu_1^2 u_3^2 \\ \frac{\mu_1 - 1}{4}(u_1^2 + u_2^2 + u_3^2) - \frac{3\mu_1 - 1}{2}(\pm u_2 u_3 \pm u_3 u_1 \pm u_1 u_2) \end{gathered} \tag{41}$$

the product of the three ambiguous signs in the last conic being 1.

These sets of conics have remarkable invariant properties. For either set the following facts are true:

The contragredient form of each conic has coefficients conjugate to those of the conic. The discriminant of each one of a set of seven has the same value. The bilinear invariant of any two (either being taken in point and the other in line form) is the same constant. This property is sufficient to determine the system. The fourteen conics are their own polars as to f^4 or ϕ^4 and are the only conics with this property.*

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* That is, each of the conics in lines has for polar as to f^4 the same conics in points.

The Ternary Covariant Quantic-Pair of Sextics.

It seems quite impracticable to treat the sextic case by a method analogous to that used for the quartic owing to the little knowledge at hand of the comitants of the sextic and the complicated nature of the invariant condition which is, in this case, the identical vanishing of a form $(ax)^3 (u\alpha)^3$ of the tenth degree in the coefficients of the sextic. But we may begin with a simplified pair of quantics and find the form they take when required to be a covariant quantic pair. Assuming then, as in all previous cases two sextics unaltered by a cyclical advance of the variables and by the changes of sign of the variables we put

$$\begin{aligned} f^6 = a\,(x_1^6 + x_2^6 + x_3^6) + 15b\ (x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2) \\ + 15c\,(x_1^4x_3^2 + x_2^4x_1^2 + x_3^4x_2^2) + 90d\,x_1^2x_2^2x_3^2 \end{aligned}$$

$$\begin{aligned} \phi^6 = \alpha\,(u_1^6 + u_2^6 + u_3^6) + 15\beta\ (u_1^4u_2^2 + u_2^4u_3^2 + u_3^4u_1^2) \\ + 15\gamma\,(u_1^4u_3^2 + u_2^4u_1^2 + u_3^4u_2^2) + 90\delta\,u_1^2u_2^2u_3^2. \end{aligned}$$

The conditions that every cubic shall have a polar cubic as to f^6 whose polar cubic as to ϕ^6 is the original cubic take the form

$$\begin{array}{lll} a\alpha + 3b\beta + 3c\gamma = \lambda & a\beta + 3b\gamma + 3c\delta = 0 & \\ 9b\beta + 9d\delta + 3c\gamma = \lambda & a\gamma + 3b\delta + 3c\beta = 0 & c\beta + 3b\delta + 3d\gamma = 0 \\ 9c\gamma + 9d\delta + 3b\beta = \lambda & c\alpha + 3b\gamma + 3d\beta = 0 & b\gamma + 3c\delta + 3d\beta = 0 \\ 36d\delta = \lambda & b\alpha + 3c\beta + 3d\gamma = 0. & \end{array}$$

The first set of four reduce to

$$2a\alpha = 45d\delta \qquad 4b\beta = 4c\gamma = 9d\delta$$

whence none of the unknowns can be zero unless $\lambda = 0$, a possibility that must be excluded. The determinant of the last set of two in the unknowns d and δ is $b\beta - c\gamma = 0$, hence the two must be proportional, *i. e.*

$$b = \sigma c,\ \gamma = \sigma\beta,\ c\beta = \sigma b\beta \text{ whence } \sigma^3 = 1.$$

Replacing c and γ by their values in terms of b and β the original system reduces to the following

$$2a\alpha = 45d\delta \qquad \frac{a}{b} = -\,3\sigma\left(1 + \sigma\frac{\delta}{\beta}\right)$$

$$4b\beta = 9d\delta \qquad \frac{\alpha}{\beta} = -3\sigma^2\left(1 + \sigma^2\frac{d}{b}\right).$$

Put $b = \beta = 1$ and we have again

$$9d\delta = 4,\ a\alpha = 10,\ a = -3\sigma(1 + \sigma\delta),\ a = -3\sigma^2(1 + \sigma^2 d).$$

If we put $a = \sigma a$, $\alpha = \sigma^2 a$, $\sigma\delta = \delta$, and $\sigma^2 d = d$ this system reduces to that obtained by taking σ as the real cube root of unity. For this last case we find easily the values of a, α, d, and δ and obtain this pair of quantics after dividing through by a and α

$$\begin{aligned} f^6 &= \Sigma x_1^6 - a_1 \Sigma x_1^4 (x_2^2 + x_3^2) + 4a_2 x_1^2 x_2^2 x_3^2 \\ \phi^6 &= \Sigma u_1^6 - a_2 \Sigma u_1^4 (u_2^2 + u_3^2) + 4a_1 u_1^2 u_2^2 u_3^2 \end{aligned} \tag{42}$$

where

$$a_1 = \frac{3(5 + i\sqrt{15})}{4} \quad a_2 = \frac{3(5 - i\sqrt{15}}{4}.$$

This is the pair of sextics (see Wiman, loc. cit.) unaltered by a ternary collineation group G_{360} and we have

(43) *A necessary condition that a pair of sextics, f^6 and ϕ^6, be unaltered by the ternary finite collineation group, G_{360} is that the sextics form a covariant quantic-pair.*

It is also extremely probable that this condition is sufficient.

The other two solutions of the above equations which arise when σ is an imaginary cube root of unity are obtained from the given solution (42) by the collineation $x_1^1 = x_1$, $x_2^1 = \sigma x_2$, $x_3^1 = \sigma^2 x_3$ and are not essentially distinct from (42).

We obtain the G_{360} in the same way as G_{168}, *i. e.*, by determining ρ in the correlation K

$$K \equiv \begin{cases} x_1 = u_1 + u_2 \\ x_2 = u_1 - u_2 \\ x_3 = \qquad\quad \rho u_3 \end{cases}$$

so that K will transform f^6 into ϕ^6. The conditions are

$$\frac{2 - 2a_1}{1} = \frac{\rho 6}{1} = \frac{30 + 2a_1}{-a_2} = \frac{-2a_1 \rho^4}{-a_2} = \frac{(-2a_1 + 4a_2)\rho^2}{-a_2} = \frac{(-12a_1 - 8a_2)\rho^2}{4a_1}$$

and the value $\pm \rho = \dfrac{\sqrt{5} + i\sqrt{3}}{2}$ satisfies all. We choose the positive sign, putting

$$\rho = \frac{\sqrt{5} + i\sqrt{3}}{2} \quad \bar{\rho} = \frac{\sqrt{5} - i\sqrt{3}}{2}$$

and transform by K the collineation, T,

$$u_1' = u_3 \qquad u_2' = u_2 \qquad u_3' = -u_1$$

which leaves ϕ^6 unaltered.

$$KTK' \equiv x' \equiv \tfrac{1}{2}\begin{pmatrix} 1 & -1 & \bar{\rho} \\ -1 & 1 & \bar{\rho} \\ -\rho & -\rho & 0 \end{pmatrix} x$$

a collineation which leaves f^6 unaltered. Interchanging the last two rows of the matrix and changing the sign of the second column we have the collineation

$$S_a \equiv x' = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & \bar{\rho} \\ -\rho & \rho & 0 \\ -1 & -1 & \bar{\rho} \end{pmatrix} x$$

The lines whose coordinates are the rows of the matrix of S_a form a perspective triangle of f^6. The triangle is unaltered by changing the sign of x_3 and also by interchanging x_2 and x_3. We obtain from it by using G_{24} only five other triangles and the corresponding matrices are

$$S_b \equiv x' = \tfrac{1}{2}\begin{pmatrix} \bar{\rho} & 1 & 1 \\ 0 & -\rho & \rho \\ \bar{\rho} & -1 & -1 \end{pmatrix} x, \quad S_c \equiv x' = \tfrac{1}{2}\begin{pmatrix} 1 & \bar{\rho} & 1 \\ \rho & 0 & -\rho \\ -1 & \bar{\rho} & -1 \end{pmatrix} x,$$

$$S_d \equiv x' = \tfrac{1}{2}\begin{pmatrix} -1 & 1 & -\bar{\rho} \\ \rho & \rho & 0 \\ 1 & -1 & -\bar{\rho} \end{pmatrix} x, \quad S_e \equiv x' = \tfrac{1}{2}\begin{pmatrix} -\bar{\rho} & -1 & 1 \\ 0 & \rho & \rho \\ -\bar{\rho} & 1 & -1 \end{pmatrix} x,$$

$$S_f \equiv x' = \tfrac{1}{2}\begin{pmatrix} 1 & -\bar{\rho} & -1 \\ \rho & 0 & \rho \\ -1 & -\bar{\rho} & 1 \end{pmatrix} x$$

S_a is a collineation of period 5. Introducing the notation

$$\varepsilon = \frac{-1 + i\sqrt{3}}{2}, \quad \varepsilon^2 = \frac{-1 - i\sqrt{3}}{2}, \quad a = \frac{-1 + \sqrt{5}}{2} \quad \bar{a} = \frac{-1 - \sqrt{5}}{2}$$

the square of S_a can be written

$$S_1 \equiv x' \equiv \tfrac{1}{2}\begin{pmatrix} -a & -\varepsilon^2 & \varepsilon\bar{a} \\ -\varepsilon^2\bar{a} & \varepsilon a & -1 \\ \varepsilon & \bar{a} & \varepsilon^2 a \end{pmatrix} x.$$

The triangle of the matrix S_1 is unaltered by the collineation $x_1 = x_2'$, $x_2 = -x_3'$, $x_3 = x_1'$. G_{24} will transform it into only 7 other triangles and we have for the corresponding matrices

$$S_2 \equiv x' = 1/2 \begin{pmatrix} \varepsilon\bar{a} & -a & -\varepsilon^2 \\ -1 & -\varepsilon^2\bar{a} & \varepsilon a \\ \varepsilon^2 a & \varepsilon & \bar{a} \end{pmatrix} x, \quad S_3 \equiv x' = 1/2 \begin{pmatrix} -\varepsilon^2 & \varepsilon\bar{a} & -a \\ \varepsilon a & -1 & -\varepsilon^2\bar{a} \\ \bar{a} & \varepsilon^2 a & \varepsilon \end{pmatrix} x,$$

$$S_4 \equiv x' = 1/2 \begin{pmatrix} -a & \varepsilon^2 & -\varepsilon\bar{a} \\ -\varepsilon^2\bar{a} & -\varepsilon a & 1 \\ \varepsilon & -\bar{a} & -\varepsilon^2 a \end{pmatrix} x \quad S_5 \equiv x' = 1/2 \begin{pmatrix} a & -\varepsilon\bar{a} & \varepsilon^2 \\ \varepsilon^2\bar{a} & 1 & -\varepsilon a \\ -\varepsilon & -\varepsilon^2 a & -\bar{a} \end{pmatrix} x,$$

$$S_6 \equiv x' = 1/2 \begin{pmatrix} -\varepsilon\bar{a} & \varepsilon^2 & a \\ 1 & -\varepsilon a & \varepsilon^2\bar{a} \\ -\varepsilon^2 a & -\bar{a} & -\varepsilon \end{pmatrix} x, \quad S_7 \equiv x' = 1/2 \begin{pmatrix} \varepsilon^2 & a & -\varepsilon\bar{a} \\ -\varepsilon a & \varepsilon^2\bar{a} & 1 \\ -\bar{a} & -\varepsilon & -\varepsilon^2 a \end{pmatrix} x,$$

$$S_8 \equiv x' = 1/2 \begin{pmatrix} a & \varepsilon\bar{a} & -\varepsilon^2 \\ \varepsilon^2\bar{a} & -1 & \varepsilon a \\ -\varepsilon & \varepsilon^2 a & \bar{a} \end{pmatrix} x. \quad \text{We add also} \quad S_0 \equiv x' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x.$$

S_2 and S_3 arise from S_1 by advancing its columns cyclically; S_4 arises from S_1 by changing the sign of the last two columns; and finally $S_5, \ldots S_8$ from $S_1, \ldots S_4$ respectively by interchanging the last two columns and changing all the signs. *The G_{360} is obtained by multiplying the G_{24} by the fifteen collineations S.*

The collineations S are all in Moore's canonical form, *i. e.*, they have determinants unity and each element is the conjugate of its minor. The same is true of the entire group if G_{24} be taken in the canonical form.

The conics unaltered by the various octahedral and ikosahedral groups in the G_{360} are easily written. We have first a set of six conics "in involution," *i. e.*, such that each is a polar to any other in the two possible ways. These conics each unaltered by an ikosahedral group are of the form

$$y_1^2 + \varepsilon y_2^2 + \varepsilon^2 y_3^2$$

$$I \qquad \frac{\rho^2 - 1}{4}(y_1^2 + y_2^2 + y_3^2) - \frac{\rho^2 + 1}{4} 2\,(y_2 y_3 + y_3 y_1 + y_1 y_2)$$

the other four being obtained by using G_{24}. The fifteen conics each unaltered by a group of one system of fifteen octahedral groups in G_{360} are of the form

$$x_1^2 + x_2^2 + x_2^3$$

$$O \qquad \frac{-\rho^2}{4}(x_1^2 + \varepsilon x_2^2 + \varepsilon^2 x_3^2) - \frac{1 + \bar{\rho}^2}{4} 2\,(x_2 x_3 + \varepsilon x_3 x_1 + \varepsilon^2 x_1 x_2,$$

$$\frac{2\bar{\rho}^2}{4} x_1^2 + \frac{2 + \rho^2}{4}(x_2^2 + x_3^2) + \frac{2 - \rho^2}{4} 2\, x_2 x_3$$

the others being obtained from the last two by the use of G_{24}.

Transforming these two sets by K and taking the point equations, which are the conjugates of the line equations, we have two new sets I' and O' corresponding to the second way of breaking up G_{360} into 6 and 15 ikosahedral and octahedral groups respectively. Those of I' are of the form

$$I' \qquad x_1^2 + x_2^2 - \bar{\rho}^2 x_3^2 + 2\,(\varepsilon - \varepsilon^2)\,x_1 x_2$$

and those of O′ of the form

$$O' \qquad x_1^2 + x_2^2 + \frac{\bar{\rho}^2}{2}\,x_3^2$$

$$\frac{4+\bar{\rho}^2}{8}(x_1^2 + x_2^2) + \frac{-4+3\bar{\rho}^2}{8}x_3^2 + \frac{\bar{\rho}\,(2-\bar{\rho}^2)}{8}\,2\,(x_1x_3 + x_2x_3) + \frac{\bar{\rho}^2}{8}x_1x_2\,.$$

The 42 *conics I, I', O, O', are the only conics which are their own second polars as to f^6 and as to ϕ^6.*

In the set I are twenty sets of three conics in involution corresponding to the twenty choices of three out of six. In the set I' are also twenty sets of three conics in involution. In O the first conic Σx_1^2 is in involution with the eight of the second form. These eight divide into four sets of two with each of which Σx_1^2 forms a set of three in involution. Hence Σx_1^2 occurs in 4 sets of three and there are then in O 15 . 4 such sets, but each counted three times. Hence

In O, O', I, I', there are 80 *sets of three conics in involution, twenty sets in each.*

Now in any linear system of three conics there are just four such sets of three conics in involution. They are the polars of the vertices of the four inflexional triangles of the cubic whose conic polar system is the given system of conics. If the cubic is harmonic two sets of three are conjugate under the G_{36} of the cubic, and the other two sets of three are also conjugate. There are in G_{360} ten G_{36} which define ten harmonic cubics which in turn define 40 sets of three conics in involution. Twenty of these are conjugate and are found in I, the other twenty are also conjugate and are found in G. We have besides the ten Hessians of the harmonic cubic which are also harmonic and define the twenty sets in I' and the twenty sets in O'.

§5. *The Forms f^{2n} and ϕ^{2n} in Higher Domain.*

Without making any pretense to generality of treatment we find, however, in the higher domains examples of covariant quantic-pairs which are essentially different from those already noted. The cases in the binary and ternary domain are all alike in that they have no absolute invariant and that the one quantic is transformable into the other by a correlation. It is easy to obtain in the quaternary domain a case in which the first property does not exist and in the quinary domain a case in which the second does not exist. Thus the pair of quartic surfaces

$$\begin{aligned} f^4 &\equiv x_1^4 + x_2^4 + x_3^4 + x_4^4 + 12h\, x_1x_2x_3x_4 \\ \phi^4 &\equiv u_1^4 + u_2^4 + u_3^4 + u_4^4 + \frac{12}{h}\, u_1u_2u_3u_4 \end{aligned} \tag{1}$$

are a covariant quantic-pair with the irrational absolute invariant h.

In the quinary domain we assume a simplified pair of quartics unaltered by the cyclical advance of the variables and their changes of sign. Take

$$\begin{aligned} f^4 &\equiv a\,(x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4) + 6b\,(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_4^2 + x_4^2x_5^2 + x_5^2x_1^2) \\ &\qquad + 6c\,(x_1^2x_3^2 + x_2^2x_3^2 + x_4^2x_5^2 + x_4^2x_1^2 + x_5^2x_2^2) \\ \phi^4 &\equiv \alpha\Sigma u_1^4 + 6\beta\Sigma u_1^2u_2^2 + 6\gamma\Sigma u_1^2u_3^2. \end{aligned} \tag{2}$$

Every quadric will have for polar as to f^4 a quadric whose polar as to ϕ^4 is the original quadric provided

$$\begin{aligned} a\alpha + 2b\beta + 2c\gamma = 4b\beta = 4c\gamma &\neq 0 \\ a\beta + b\alpha + c\beta + b\gamma + c\gamma &= 0 \\ a\gamma + c\alpha + c\beta + b\gamma + b\beta &= 0 \end{aligned} \tag{3}$$

From the first equation we have $a\alpha = 0$.

Suppose $\alpha = 0 \quad a \neq 0$. The solution is

$$a = 1 \quad b = c = \mp 1/3 \quad \alpha = 0 \quad \beta = \gamma$$

and we have the covariant quantic-pair

$$\begin{aligned} f^4 &\equiv \Sigma x_1^4 - 2\Sigma x_1^2\,(x_2^2 + x_3^2) \\ \phi^4 &\equiv \Sigma u_1^2\,(u_2^2 + u_3^2) \end{aligned} \tag{4}$$

f^4 has 30 double points of the type 0, 0, 0, 1, ± 1 while ϕ^4 has only five double flats of the type 0, 0, 0, 0, 1.

We have then the curious example of two quantics each the same covariant of the other and yet not transformable into each other.

If we solve the system of equations (3) more symmetrically by assuming both $a = 0$ and $\alpha = 0$ we have the quantic-pair

$$\begin{aligned} f^4 &\equiv \Sigma x_1^2 x_2^2 + \rho \Sigma x_1^2 x_3^2 \\ \phi^4 &\equiv \Sigma u_1^2 u_2^2 + \rho^2 \Sigma u_1^2 u_3^2. \end{aligned} \qquad \rho = \frac{-1 \pm i\sqrt{3}}{2} \tag{5}$$

We shall close with a discussion, in the next paragraph, of the covariant quantic-pair (5) which has properties entirely analogous to the pairs found in the binary and ternary domain.

§6. *The Finite Collineation Group, G_{25920}, in Five Variables.*[*]

We see that the quantics f^4 and ϕ^4 in (5) §5 are unaltered by the collineations

$$S \equiv \begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = x_5 \\ x_5' = x_1 \end{cases} \qquad T \equiv \begin{cases} x_1' = x_1 \\ x_2' = \rho^2 x_3 \\ x_3' = \rho x_2 \\ x_4' = \rho x_5 \\ x_5' = \rho^2 x_4 \end{cases} \qquad U \equiv \begin{cases} x_1' = x_1 \\ x_2' = x_5 \\ x_3' = x_4 \\ x_4' = x_3 \\ x_5' = x_2 \end{cases}$$

S, T, and U generate an ikosahedral group G_{60}, which permutes evenly the five points and five flats of the pentahedron of reference, P_0. This G_{60} combined with the G_{16} made up of the changes of sign of the variables gives rise to a G_{960} which leaves P_0 unaltered. There are no other collineations which do not alter f^4 and P_0.

The vertices of P_0 are double points, *DP's* of f^4 and the flats of P_0 are double flats, *DE's*, of ϕ^4. We find without difficulty that f^4 has just 45 *DP*, 9 of the form

$$1,\ 0,\ 0,\ 0,\ 0 \qquad 0,\ \pm 1,\ \pm \rho,\ \pm \rho,\ \pm 1 \tag{1}$$

the 36 others being obtained from these by the use of S.

Also ϕ^4 has 45 *DE* whose coordinates are the conjugates of the coordinates of the 45 *DP*. The *DP* are all of the same type—the polar of each as to f^4 breaks up into a flat, the conjugate *DE*, and a hyper-cone with double point at the *DP* in question. The dual statement holds for the 45 *DE*. We have then a correspondence between the *DP* and *DE*, each corresponding to its conjugate. There is no trouble in verifying from (1) that

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[*] A complete enumeration of the subgroups of this group is given by L. Dickson, Transactions American Mathematical Society, Vol. 5, 1904, p. 126. Further references are given there, the articles by Burkhardt being especially pertinent to this account.

Each DE passes through 12 *of the* 45 *DP and each DP lies in* 12 *of the* 45 *DE.* Every collineation that leaves a *DE* unaltered must leave the corresponding *DP* unaltered and hence the twelve *DE* through it. In the *DE*, $x_1 = 0$, we have twelve points dividing into 3 tetrahedra namely.*

$$\begin{array}{cccc@{\qquad}cccc@{\qquad}cccc}
1 & 0 & 0 & 0 & 1 & \rho & \rho & 1 & 1 & \rho & \rho & -1 \\
0 & 1 & 0 & 0 & 1 & \rho & -\rho & -1 & 1 & \rho & -\rho & 1 \\
0 & 0 & 1 & 0 & 1 & -\rho & \rho & -1 & 1 & -\rho & \rho & 1 \\
0 & 0 & 0 & 0 & 1 & -\rho & -\rho & 1 & -1 & \rho & \rho & 1
\end{array} \tag{2}$$

And the twelve *DE* through $u_1 = 0$ cut out on $\chi_1 = 0$ the twelve planes (2′) of the above three tetrahedra, whose coordinates are the conjugates of the coordinates of the points (2). The three tetrahedra (2) form the well known desmic system of tetrahedra, which is unaltered by a group of 576 collineations. If we require that f^4 also be unaltered by the collineations then the group can permute the variables only in the even ways and its order is reduced to 288. However, a change of sign of x_1 being added to the group the order is restored to 576. This G_{576} is generated by the tetrahedral subgroup of G_{60} which leaves $x_1 = 0$ unaltered, by the G_{16} of the changes of sign of the variables and by the collineation S_a of period 3, whose square is S_{a}

$$S_a \equiv 2x' = \begin{array}{c} \\ \\ \rho \\ \rho \\ \\ \end{array}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & \rho^2 & \rho^2 & 1 \\ 0 & 1 & \rho^2 & -\rho^2 & -1 \\ 0 & 1 & -\rho^2 & \rho^2 & -1 \\ 0 & -1 & \rho^2 & \rho^2 & -1 \end{pmatrix} x \qquad S_{\mathrm{a}} \equiv 2x' = \begin{array}{c} \\ \\ \rho \\ \rho \\ - \end{array}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & \rho^2 & \rho^2 & -1 \\ 0 & 1 & \rho^2 & -\rho^2 & 1 \\ 0 & 1 & -\rho^2 & \rho^2 & 1 \\ 0 & -1 & \rho^2 & \rho^2 & 1 \end{pmatrix}$$

The factors outside the matrix are supposed to multiply each element. In the row opposite, S_a and S_{a} permute the three tetrahedra (2) cyclically. They are of determinant unity and in the usual canonical form. They leave the configuration of 45 *DP* unaltered and therefore f^4 as well. Hence

(3) *Of the group of collineations which leaves* f^4 *unaltered there is a* G_{576} *which leaves one of the* 45 *DP unaltered.* f^4 *then is automorphic under a group of order* 45. 576 = 25920.

Since the pentahedron of reference P_0 is unaltered by a G_{960} we should expect it to be transformed by G_{25920} into $\frac{25920}{960}$ or 27 others. The rows of the matrices of S_a and S_{a} determine two others, P_a and P_{a}. Both P_a and P_{a} contain the *DE*, $x_1 = 0$, as well as P_0. If each *DE* is contained in 3 pentahedra we should have again $\frac{45\,.\,3}{5}=27$ pentahedra. These pentahedra are obtained from the matrices of S_a, S_α and S_0 where

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* For a complete geometrical discussion of this system and the G_{288} see J. Feder, Math. Annalen, Vol. 47, p. 375.

$$S_0 \equiv x' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} x$$

and the matrices of twenty-four other collineations derived as follows:— Transform the collineations S_a and S_α by the collineation S of G_{60}. We have

$$S_b \equiv 2x' = \begin{array}{r} \\ \\ \\ \rho \\ \rho \end{array}\begin{pmatrix} -1 & 0 & -1 & \rho^2 & \rho^2 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & \rho^2 & \rho^2 \\ -1 & 0 & 1 & \rho^2 & -\rho^2 \\ -1 & 0 & 1 & -\rho^2 & \rho^2 \end{pmatrix} x \qquad S_\beta \equiv 2x' = \begin{array}{r} - \\ \\ \\ \rho \\ \rho \end{array}\begin{pmatrix} 1 & 0 & -1 & \rho^2 & \rho^2 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 1 & \rho^2 & \rho^2 \\ 1 & 0 & 1 & \rho^2 & -\rho^2 \\ 1 & 0 & 1 & -\rho^2 & \rho^2 \end{pmatrix} x$$

$$S_c \equiv 2x' = \begin{array}{r} \rho \\ \\ \\ \\ \rho \end{array}\begin{pmatrix} \rho^2 & -1 & 0 & 1 & -\rho^2 \\ \rho^2 & -1 & 0 & -1 & \rho^2 \\ 0 & 0 & 2 & 0 & 0 \\ \rho^2 & 1 & 0 & 1 & \rho^2 \\ -\rho^2 & -1 & 0 & 1 & \rho^2 \end{pmatrix} x \qquad S_\gamma \equiv 2x' = \begin{array}{r} \rho \\ - \\ \\ \\ \rho \end{array}\begin{pmatrix} \rho^2 & 1 & 0 & 1 & -\rho^2 \\ \rho^2 & 1 & 0 & -1 & \rho^2 \\ 0 & 0 & 2 & 0 & 0 \\ \rho^2 & -1 & 0 & 1 & \rho^2 \\ -\rho^2 & 1 & 0 & 1 & \rho^2 \end{pmatrix} x$$

$$S_d \equiv 2x' = \begin{array}{r} \rho \\ \rho \\ \\ \\ \\ \end{array}\begin{pmatrix} \rho^2 & -\rho^2 & -1 & 0 & 1 \\ -\rho^2 & \rho^2 & -1 & 0 & 1 \\ \rho^2 & \rho^2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ \rho^2 & \rho^2 & 1 & 0 & 1 \end{pmatrix} x \qquad S_\delta \equiv 2x' = \begin{array}{r} \rho \\ \rho \\ - \\ \\ \\ \end{array}\begin{pmatrix} \rho^2 & -\rho^2 & 1 & 0 & 1 \\ -\rho^2 & \rho^2 & 1 & 0 & 1 \\ \rho^2 & \rho^2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ \rho^2 & \rho^2 & -1 & 0 & 1 \end{pmatrix} x$$

$$S_e \equiv 2x' = \begin{array}{r} \\ \rho \\ \rho \\ \\ \\ \end{array}\begin{pmatrix} 1 & \rho^2 & \rho^2 & 1 & 0 \\ 1 & \rho^2 & -\rho^2 & -1 & 0 \\ 1 & -\rho^2 & \rho^2 & -1 & 0 \\ -1 & \rho^2 & \rho^2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} x \qquad S_\epsilon \equiv 2x' = \begin{pmatrix} 1 & \rho^2 & \rho^2 & -1 & 0 \\ 1 & \rho^2 & -\rho^2 & 1 & 0 \\ 1 & -\rho^2 & \rho^2 & 1 & 0 \\ -1 & \rho^2 & \rho^2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} x$$

In S_b change the signs of the first and third columns of the matrix and call the collineation S'_b. With the collineation S^2TS^3 of G_{60} form the combination $S^2TS^2S'_bS_\alpha$. The result is

$$S_1 \equiv 2\rho x' = \begin{pmatrix} 0 & 1 & \rho^2 & \rho^2 & 1 \\ \rho^2 & \rho^2 & 1 & 0 & 1 \\ 1 & 0 & 1 & \rho^2 & \rho^2 \\ 1 & \rho^2 & \rho^2 & 1 & 0 \\ \rho^2 & 1 & 0 & 1 & \rho^2 \end{pmatrix}$$

The matrix of this is of different type from the eleven already found and we find by changing the signs of the columns in all possible ways 15 new pentahedra which with that of S_1 we shall name $P_1, P_2, \ldots, P_{16}$ and the corresponding collineations $S_1, \ldots S_{16}$. Then $S_1 \,.\, G_{16} = S_1, S_2, \ldots S_{16}$. These results may be incorporated in the theorem:

(4) *There are 27 pentahedra whose vertices belong to the 45 DP and whose flats belong to the DE. Each of the five flats contains besides the four vertices eight DP, each of the five points lies on eight DE in addition to the four flats of the pentahedron, i. e., the five flats contain the 45 DP and the five vertices lie on the 45 DE. The* G_{25920} *contains one system of 27 conjugate subgroups of order 960 each subgroup having an invariant pentahedron.*

The fact that f^4 and ϕ^4 have different forms when referred to the same pentahedron, P_0, does not lead to the conclusion, as one might suppose, that there are two systems of 27 conjugate G_{960}. For the correlation, K,

$$x_1 = u_1, \quad x_2 = \rho^2 u_5, \quad x_3 = \rho u_3, \quad x_4 = \rho u_4, \quad x_5 = \rho^2 u_2$$

transforms f^4 into ϕ^4 without altering P_0. It differs from the collineations that do not alter P_0 in that the vertices of P_0 are carried into an *odd* permutation of the flats of P_0.

We see at once that the G_{60} generated by S, T, and U, since it contains no changes of sign of the variables, must also effect the even permutations of the flats of P_1. P_0 and P_1 have no common vertices. The same is true of P_0 and P_2, P_0 and $P_3, \ldots,$ P_0 and P_{16}. But since P_1 is transformed into P_ι ($\iota = 1, \ldots 16$) by G_{16} we find for each pair P_0 and P_ι a different G_{60} the transform of the original G_{60} by some collineation of G_{16}. Hence

(5) G_{25820} *contains a system of* 1/2 27.16 *conjugate* G_{60}, *any member of the system being defined as the group which leaves each of two pentahedra, having no common vertex, unaltered.*

The collineation S_1 is of period 4. Its square is T of the original G_{60}. Hence S_1 permutes P_0 and P_1 and with G_{60} generates a G_{120} isomorphic with the symmetric group of five things. One such set of five things is P_a, P_b, P_c, P_d, P_e which are the only pentahedra each containing a flat of P_0 and a flat of P_1. If we call P_2, P_4, P_6, P_8, P_{10} the pentahedra which arise from P_1 by changing the sign of the first, second, . . . , fifth columns respectively we see that P_2, P_4, P_6, P_8, P_{10} each contains one of the flats of P_1 and that P_α, P_β, P_γ, P_δ, P_ϵ each contains one of the flats of P_0. Hence the first set of five pentahedra is permuted evenly by G_{60} and the same is true of the second set while the odd substitutions of G_{120} permute the two sets.

(6) *Hence in* G_{25920} *there is a system of* 27.8 *conjugate* G_{120} *each of which is defined as the group which does not alter a set of two pentuhedra having no common vertex.*

We consider now the following group of twelve pentahedra which divide into two sets of six, each in one set being paired with a "partner" in the other,

$$D_6 \equiv \begin{Bmatrix} P_0, & P_2, & P_4, & P_6, & P_4, & P_{10} \\ P_1, & P_\alpha, & P_\beta, & P_\gamma, & P_\delta, & P_\epsilon \end{Bmatrix}$$

Each P in the upper line has a flat and vertex in common with every P in the lower line except its partner. The choice of any pair determines the five others. The pair P_0, P_1 being unaltered by a G_{120} which simply permutes the other five pairs we can conclude that the entire configuration D_6 is unaltered by a G_{720}. We can easily construct a unique quadric which admits the original G_{60} namely

$$Q \equiv \rho^2 (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) + \rho (x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2) \qquad (7)$$

But this passes through the vertices not only of P_0 but also of all the other pentahedra of D_6. The equation in flats is the conjugate of (7) whence it touches all the flats of D_6. (7) must be unaltered therefore by the entire G_{720} and be transformed by G_{25920} into only 36 quadrics. 16 of the 36 are obtained from (7) by using G_{16}. Forming (7) for the pentahedron P_α we have

$$Q \equiv \rho x_3^2 + x_4^2 + \rho x_5^2 \pm 2\rho^2 x_1x_2 \qquad (8)$$

and applying T of G_{60} to this we have

$$Q \equiv x_2^2 + \rho^2 (x_4^2 + x_5^2) \pm 2\rho x_1x_3 \qquad (9)$$

From the four quadrics (8) and (9) we obtain by the use of S the twenty remaining quadrics Q.

(10) *There are then thirty-six quadrics Q, each of which passes through the thirty vertices and is touched by the thirty flats of a D_6. The flat and vertex forming an element of Q occurs in two pairs of D_6. There are in G_{25920} 36 conjugate G_{720} one corresponding to each Q and its inscribed D_6.*

Another noteworthy configuration associated with f^4 and ϕ^4 is a system of points N and flats F. One of the flats F has the equation

$$x_1 + \rho^2 x_2 + \rho^3 x_5 = 0 \qquad (11)$$

and is defined by the property it possesses of passing through 18 of the 45 DP.

To find the number of the F we observe that through one DP, *e.g.*, 1, 0, 0, 0, 0 pass at least 16 F, namely the four

$$\begin{array}{ccccc} 0 & \rho^2 & 1 & \rho^2 & 0 \\ 0 & 0 & \rho^2 & 1 & \rho^2 \\ 0 & \rho & \rho & \rho & 1 \\ 0 & 1 & 0 & \rho & \rho \end{array}$$

and the twelve obtained from these by changing the signs of their coordinates. Each of these sixteen passes through a "line of the desmic system of tetrahedra" in $x_1 = 0$, *i. e.*, a line of the system containing one vertex of each of the three tetrahedra. There are just 16 such lines, one in each of the 16 F. The line and DP do not fix the F, but it can easily be verified that through the DP and the line there can be only one F that contains 18 DP. Hence the number of the flats F is $\frac{16.45}{18} = 40$. They may all be obtained from the sixteen mentioned by the use of S.

(12) *The* 45 *DP lie by* 18*'s on* 40 *flats F; the* 45 *DE pass by* 18*'s through* 40 *points N whose coordinates are conjugate to those of F. No N lies on an F; no F passes through an N. In* G_{25920} *is a system of* 40 *conjugate* $G_{8.216}$, *each having for invariant an F and an N.*

The G_{25920} has been given * as a group in four variables containing as subgroups the group G_{216} in three variables. In our case the subgroup which leaves an F invariant appears in that three dimensional space as a collineation group projectively different from the ternary G_{216}. The subgroup in F is indeed the group $G_{8.216}$ of the three dimensional tetrahedral cubic $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$. We verify this as follows:—Assuming a flat F as invariant the remaining 39 break up into a set of 12 and a set of 27. The set of 12 breaks up into 4 sets of three, each one of a set of three cutting F in the same plane and the 12 therefore giving rise to a tetrahedron on F. Hence

(13) *The* 18 *DP on F lie by* 9*'s on the four planes of a tetrahedron in F; two planes pass through each DP, hence on each edge of the tetrahedron lie three of the DP. The dual of this is true for an N.*

We also have that

(14) *The* 40 *F meet by* 4*'s in forty planes H. The* 40 *N lie by* 4*'s on forty lines L.*

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* Maschke, Math. Annalen, p. 317. Further references are given there.

The remaining 27 F above mentioned are thus accounted for: On each of three edges of the tetrahedron in F that meet in a point are three DP which determine 27 planes, the planes of intersection of the 27 F with the invariant F. This is then exactly the configuration determined by the tetrahedral cubic surface.

For later use we may note also that there is through every pair of points DP which do not form vertices of the same P a line containing a third DP contained in no pentahedra containing the first two DP. We may call such a set of three a "line-triad." As an example we have that the three points

$$\begin{array}{lll} 1, 0, 0, 0, 0 & \text{contained in} & P_a,\ P_o,\ P_\sigma \\ 1, 0, 1, \rho, \rho & \text{“} \quad \text{“} & P_1,\ P_b,\ P_4 \\ -1, 0, 1, \rho, \rho & \text{“} \quad \text{“} & P_2,\ P_\beta,\ P_3 \end{array} \tag{15}$$

lie on a line. The 9 pentahedra are determined however by a second "line-triad" for

$$\begin{array}{lllll l l} 0, & 1, & 0, & 0, & 0 & \text{is contained in} & P_0,\ P_b,\ P_\beta \\ 0, & 1, & \rho, & \rho, & 1 & \text{“} \quad \text{“} \quad \text{“} & P_a,\ P_1,\ P_2 \\ 0, & -1, & \rho, & \rho, & 1 & \text{“} \quad \text{“} \quad \text{“} & P_a,\ P_4,\ P_3 \end{array} \tag{16}$$

We shall say that one line-triad determines uniquely its "conjugate line-triad." The two lines are skew and determine therefore a flat, one of the forty F, $\rho^2 x_3 + x_4 + \rho^2 x_5 = 0$. In F the two lines are opposite edges of the tetrahedron [see (13)] and hence a pair of conjugate line-triads determine uniquely two other pairs which are reciprocally related to it. We find in all 240 triads since there are forty F. But it is clear also that through a DP pass 16 line-triads whence their number is $\frac{45.16}{3} = 240$.

(17) *There are then* 40 *sets of reciprocally related three pairs of conjugate line-triads.*

We have just seen that one line-triad, l, determines a conjugate and the two a flat F. In any flat F are 42 l's, the six edges of the tetrahedron of (13) and 9 in each of the faces. Hence through each l pass $\frac{42.40}{240} = 7$ F, a number that is easily recognized for each of the 42 lines in an F. Thus through the l of (15) pass the flats F

$$\begin{array}{ccccc} 0 & 0 & \rho^2 & 1 & \rho^2 \\ \hline 0 & \rho^2 & -1 & \rho^2 & 0 \\ 0 & \rho & \rho & 0 & -1 \\ 0 & 1 & 0 & -\rho & \rho \\ \hline 0 & \rho^2 & 1 & -\rho^2 & 0 \\ 0 & \rho & -\rho & 0 & 1 \\ 0 & 1 & 0 & \rho & -\rho \\ \hline \end{array} \tag{18}$$

We note that the first is singled out by lying in a pencil with each of the two sets of three. This first is exactly the F determined by the pair of conjugate l, (15) and (16). Hence

(19) *In each F are 42 l; through each l are 7 F which form two pencils of 4, the two pencils having a common F. Six l determine the same F, the six forming the edges of a tetrahedron in F. The forty tetrahedra arising from the forty F contain as edges all the 240 lines.*

Besides the 40 F we have in (14) a set of 40 conjugate planes H, the faces of the 40 tetrahedra in F, each face occuring in 4 F that form a pencil. Each of the 40 H are unaltered by a $G_{3.216}$ and we wish to find whether it is contained in the set of conjugate $G_{3.216}$ already found. But the $G_{3.216}$ of (12) has for invariant a tetrahedron in the invariant F and it is not difficult to show that the remaining 39 F under the subgroup which leaves a face of the tetrahedron unaltered break up into a set of three, a set of 9, and a set of 27 F, the members of any set being conjugate. None of the 39 are unaltered whence the group that permutes four F that form a pencil (*i. e.*, that has for invariant an H) has no invariant F and is distinct from the $G_{3.216}$ of (12). Since H is common to 4 F and in each F is a tetrahedron peculiar to itself it follows that the 12 l in a face of the tetrahedron of an F lie in 4 triangles, each triangle belonging to the tetrahedron of one of the 4 F. Hence in H we have the configuration of triangles associated with a syzygetic pencil of cubics.

(20) *The G_{25920} contains two different systems of forty $G_{3.216}$. One is the system described in (12); the other is the system which has for invariant configurations the forty planes H or the forty lines L. In an invariant plane H, the invariant configuration is the same as that of the ternary G_{216}.*

We have obtained above the subgroups of G_{25920} of smaller indices. It is not intended to attempt here a complete classification of the various subgroups of G_{25920}. But it is clear that for such a purpose the above representation of the group is well suited. The variety of geometrical figures that belong to certain subgroups and that are of simple projective character is very great. For example, besides the flats F through 18 DP and the flats DE through 12 DP, there are flats of the type $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ through 10 DP, flats of the type $x_2 + x_5 = 0$ through 7 DP, and flats of the type $x_2 + x_3 + x_4 = 0$ through 6 DP. Also the 40 points N, dual to the F have barely been mentioned.

That G_{25920}, abstractly considered, is isomorphic with the group of the equations of the 27 lines of a cubic surface has long been known.* A few simple correspondence principles permit of deriving the more important properties of these 27 lines from the configuration described above. These are :—

(*a*) *To a line* λ *of the cubic surface corresponds a pentahedron* P.

(*b*) *To two lines* λ *which intersect* (*lie in a plane* ε) *correspond two pentahedra* P *with a common vertex, a* DP.

With the aid of (*a*) and (*b*) the collection of facts concerning the 27 λ given in Pascal's Repertorium† can be obtained without difficulty from the properties of the 27 P. Thus from the fact that each P has 5 vertices and each vertex belongs to 3 P we see that

1° *There are* 45 ε *each containing* 3λ. *Each* λ *occurs in* 5 ε.

P_0 has a vertex in common with $P_a, \ldots, P_b$ and $P_\alpha, \ldots, P_\epsilon$ but not with $P_1, \ldots, P_{16}$. Hence

2° *Each* λ *cuts* 10 *other* λ *and does not cut the remaining* 16. *Two* λ *that do not cut will both be cut by* 5λ, *each be cut by* 5 *other* λ, *and neither be cut by the remaining* 10λ.

The latter part of this is evident from the fact that P_0 and P_1 have a vertex in common with $P_a, \ldots, P_e$, that P_0 has one in common with $P_\alpha, \ldots, P_\epsilon$, and that P_1 has one in common with $P_2, P_4, P_6, P_8, P_{10}$. From (6) we have

3° *There are* 216 *skew-pairs* λ.

The existence of the D_6 (p. 50) shows that

4° *There are* 36 "*double-sixes.*"

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* Jordan, Traité Des Substitutions, p. 316.

† *E.* Pascal, Repertorium Der Höheren Mathematik, II, p. 284.

From the double sixes we can pass at once to the various systems of non-cutting lines mentioned by Pascal. That the quadric Q (7) has the same form as fifteen others but a different form from that of the twenty (8) and (9) leads to the theorem

5° *A double six has* 4λ *in common with each of* 15 *others and* 6λ *in common with each of* 20 *others.*

Two DP not contained in a P lie on a line with a third DP; this line determines another such that the nine P having vertices at the 3 DP are the same for the two sets of three. There are 120 such line pairs. Each pair determines an F and in F is one of the three pairs of opposite edges of a tetrahedron. The 18 points on the six lines are the vertices of 54 P, each of the 27 P occurring twice (see (15), (16), (17), (18), (19)). The translation of the above results in

6° *Two* ε *having no* λ *common to each determine a third* ε, *no two of the three having a* λ *common to each;* 9λ *in the* 3ε *are contained similarly in a second set of three* ε. *The two sets of* 3ε *are called a "pair of conjugate trihedra." There are* 120 *such pairs. Any one pair of conjugate trihedra determines two other pairs such that the complex of three pairs contains the* 27λ. *There are* 40 *such complexes.*

Finally, the considerations which led to the second system of 40 conjugate $G_{3.216}$ may be translated as follows:—

(21) *A complex of conjugate trihedra-pairs contains* 18ε. *These* 18ε *may be divided in four ways into two sets of* 9ε, *a set A and a set B such that the set A contains the* 27λ *and the set B contains the* 27λ. *There are* 160 *sets A and* 160 *sets B. The* 160 *sets A are each common to a group of* 4 *complexes. There are* 40 *"complex-groups of four,"* i. e., 4 *complexes having a set A of* 9ε *common. However, no two complexes have a set B of* 9ε *common.*

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On Some Cases of Motion of a Solid in Infinite Liquid.

By G. Kolosoff.

The present paper contains some elliptic-function solutions of the motion under no force of a solid in infinite liquid of the same character as considered by Mr. A. G. Greenhill in Vol. XX and XXVIII of the American Journal of Mathematics, but for a body of different external shape.

(I) The external shape of the body is that considered by Clebsch in Mathematische Annalen III with the form of the kinetic energy[*]

$$2T = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + b_1y_1^2 + b_2y_2^2 + b_3y_3^2$$

where

$$a_1\left(\frac{1}{b_2} - \frac{1}{b_3}\right) + a_2\left(\frac{1}{b_3} - \frac{1}{b_1}\right) + a_3\left(\frac{1}{b_1} - \frac{1}{b_2}\right) = 0. \tag{1}$$

With the six dynamical equations:

$$\frac{dx_1}{dt} = -x_3\frac{\partial T}{\partial y_2} + x_2\frac{\partial T}{\partial y_3}, \quad \frac{dy_1}{dt} = -x_3\frac{\partial T}{\partial x_2} + x_2\frac{\partial T}{\partial x_3} - y_3\frac{\partial T}{\partial y_2} + y_2\frac{\partial T}{\partial y_3}$$

. . . .

we arrive immediately at the three integrals:

$$\left.\begin{array}{ll} 2T = const. = l & x_1y_1 + x_2y_2 + x_3y_3 = const. = m \\ x_1^2 + x_2^2 + x_3^2 = const. = n & \end{array}\right\} \tag{2}$$

and the fourth integral of Clebsch:

$$b_1a_1y_1^2 + b_2a_2y_2^2 + b_3a_3y_3^2 - (a_2a_3x_1^2 + a_3a_1x_2^2 + a_1a_2x_3^2) = const. = k. \tag{3}$$

In particular cases $m = 0$ and $b_1 = b_2 = b_3$ this motion is expressible by the double-theta hyperelliptic function.[†] The most general case was reduced by F. Kötter to the case $b_1 = b_2 = b_3$, but his investigation contained mistakes as shown by the investigations of the Russian mathematicians A. Liapounoff and W. Stekloff.[*]

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[*] The notation employed is that of the paper of Mr. A. G. Greenhill.

[†] H. Weber, Mathematische Annalen XIV. F. Kötter, Crelle, t. 109.

Let us suppose that the constants in the expression of the kinetic energy satisfy besides (1) also the conditions:

$$\frac{1}{b_3}=\frac{1}{b_1}+\frac{1}{b_2}{}^{\dagger} \quad (b_3 < b_1 \text{ and } b_3 < b_2).$$

Then $(a_3-a_2)\,b_1=(a_3-a_1)\,b_2$ and $(a_1-a_2)\,b_2=(a_3-a_2)\,(b_2-b_1)$.
From the integrals (2) and (3) we have:

$$b_2\,(a_3-a_1)\,(x_1^2+x_2^2)-(b_2^2y_2^2+b_1^2y_1^2)=\frac{b_2}{a_3-a_2}\Big(la_3-k-n\,(a_3^2+a_1a_2)\Big).$$

Let us suppose

$$la_3=k+n\,(a_3^2+a_1a_2).$$

The differential equations admit then a fractional integral:

$$\frac{b_1x_1y_1+b_2x_2y_2}{(a_3-a_2)\,x_2^2-b_1y_1^2}=const.=\Gamma. \tag{3*}$$

Besides this integral the fundamental integrals give us the integral

$$x_3-\frac{a_3-a_2}{a_2-a_1}\,\Gamma\,y_3=const.=\Gamma_1. \tag{4}$$

Bearing in mind that

$$\varepsilon^2\,(x_1^2+x_2^2)^2-(b_1x_1y_1+b_2x_2y_2)^2=(b_2y_2x_1-b_1y_1x_2)^2$$

where

$$\varepsilon=+\sqrt{b_2\,(a_3-a_1)}=+\sqrt{b_1\,(a_3-a_2)},$$

and taking into consideration the integral

$$x_1^2+x_2^2+x_3^2=const.=n$$

we deduce

$$\left(\frac{dx_3}{dt}\right)^2=(b_2y_2x_1-b_1y_1x_2)^2=\varepsilon^2\,(n-x_3^2)^2-\Gamma^2\Big((a_3-a_2)\,x_2^2-b_1y_1^2\Big)^2. \tag{5}$$

---

[*] W. Stekloff. On the motion of solid in infinite liquid. Charcow, 1893 (in Russian language).

[†] The cases $\frac{1}{b_1}=\frac{1}{b_2}+\frac{1}{b_3}$ and $\frac{1}{b_2}=\frac{1}{b_1}+\frac{1}{b_3}$ can be treated similarly.

But from the integrals (2) and (3) we deduce

$$(a_1 - a_2)\, b_2 x_2^2 + (b_1 - b_2)\, b_1 y_1^2 = b_2 b_3 y_3^2 + \Gamma_2, \text{ where } \Gamma_2 = n b_2 a_3 - b_2 k,$$

and (5) takes the form

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2 (n - x_3^2)^2 - \Gamma^2 \frac{(b_2 b_3 y_3^2 + \Gamma_2)^2}{(b_2 - b_1)^2}, \tag{6}$$

since

$$(a_3 - a_2)(b_2 - b_1) = (a_1 - a_2)\, b_2 \tag{7}$$

and

$$(a_3 - a_2)\, x_2^2 - b_1 y_1^2 = \frac{(a_1 - a_2)\, b_2}{b_2 - b_1} x_2^2 - b_1 y_1^2 = \frac{1}{b_2 - b_1}\Big((a_1 - a_2)\, b_2 x_2^2 + b_1 (b_1 - b_2) y_1^2\Big)$$

$$= \frac{1}{b_2 - b_1}(b_2 b_3 y_3^2 + \Gamma_2),$$

and from (4), (6):

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2 (n - x_3^2)^2 - \frac{\Gamma^2}{(b_2 - b_1)^2}(b_2 b_3 y_3^2 + \Gamma_2)^2 =$$

$$= \varepsilon^2 \left\{ (n - x_3^2)^2 - \frac{\Gamma^2}{(b_2 - b_1)^2 \varepsilon^2}\left(b_2 b_3 \frac{(a_2 - a_1)^2}{\Gamma^2 (a_3 - a_2)^2}\,(x_3 - \Gamma_1)^2 + \Gamma_2\right)^2 \right\},$$

and therefore

$$\int \frac{dx_3}{\sqrt{(n - x_3^2)^2 - \frac{\Gamma^2}{(b_2 - b_1)^2\, \varepsilon^2}\left(b_2 b_3 \frac{(a_2 - a_1)^2}{\Gamma^2 (a_3 - a_2)^2}\,(x_3 - \Gamma_1)^2 + \Gamma_2\right)^2}} = \varepsilon t + const.,$$

or

$$\int \frac{dx_3}{\sqrt{(n - x_3^2)^2 - (\alpha + \beta x_3 + \gamma x_3^2)^2}} = \varepsilon t + const.,$$

where α, β, γ are constants.

From this we can conclude that the problem may be reduced to elliptic functions.

I have shown* that the motion in this case can be resolved into two motions à la Poinsot and that the elements of the motion can be expressed by Weierstrassian Sigma Functions.

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* In my Russian paper "On Some Modifications of the Hamilton's Principle" this solution is developed and discussed. A short account of this paper is given by Lampe in "Forschritte der Mathematik," 1903.

As individual cases of this theory we may regard and reduce to elliptic functions the following cases:

$$
\begin{array}{lll}
\text{(A)} & y_1 b_1 = \varepsilon x_2, & y_2 b_2 = \varepsilon x_1 \\
\text{(B)} & y_1 b_1 = -\varepsilon x_2, & y_2 b_2 = -\varepsilon x_1 \\
\text{(C)} & y_1 b_1 = \varepsilon x_1, & y_2 b_2 = -\varepsilon x_2 \\
\text{(D)} & y_1 b_1 = -\varepsilon x_1, & y_2 b_2 = \varepsilon x_2.
\end{array}
$$

We may deduce these solutions immediately from the six differential equations without using the integral (3*). Let us suppose (A) $y_1 b_1 = \varepsilon x_2$, $y_2 b_2 = \varepsilon x_1$. The first three differential equations of the problem take the form:

$$
\left.
\begin{aligned}
\frac{dx_1}{dt} &= b_3 y_3 x_2 - \varepsilon x_1 x_3 \\
\frac{dx_2}{dt} &= \varepsilon x_2 x_3 - b_3 y_3 x_1 \\
\frac{dx_3}{dt} &= \varepsilon (x_1^2 - x_2^2)
\end{aligned}
\right\} \qquad (8)
$$

The 6th equation gives

$$y_3 = const. = \omega,$$

the 4th and the 5th become

$$\frac{dx_2}{dt} = \frac{b_1(a_3 - a_2)}{\varepsilon} x_1 x_3 + \frac{(b_3 - b_2) b_1}{b_2} x_1 y_3;$$

$$\frac{dx_1}{dt} = \frac{b_2(a_1 - a_3)}{\varepsilon} x_3 x_1 + \frac{(b_1 - b_3) b_2}{b_1} x_2 y_3,$$

and coincide with the first two equations (8), by (7) and by the relation

$$(a_3 - a_1) b_2 = (a_3 - a_2) b_1 = \varepsilon^2.$$

The integrals (2) take the form

$$x_1^2 + x_2^2 + x_3^2 = n$$

$$x_1 x_2 + \frac{b_3}{\varepsilon} \omega x_3 = \frac{m b_3}{\varepsilon},$$

and we have

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2 (n - x_3^2)^2 - 4 b_3^2 (m - \omega x_3)^2.$$

Therefore the problem is reduced to elliptic functions. In the same manner we can investigate the case B. Let us suppose (C) $y_1 b_1 = \varepsilon x_1 \quad y_2 b_2 = -\varepsilon x_2$. The differential equations of motion take the form:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= b_3 y_3 x_2 + \varepsilon x_2 x_3 \\ \frac{dx_2}{dt} &= \varepsilon x_1 x_3 - b_3 y_3 x_1 \\ \frac{dx_3}{dt} &= -2\varepsilon x_1 x_2 \\ \frac{dy_3}{dt} &= 2(a_2 - a_1) x_1 x_2 \end{aligned}\right\} . \tag{9}$$

The other two equations coincide with (9) in the same way as in case A.

The integrals (2) and (3) are reduced to

$$\left.\begin{aligned} & x_1^2 + x_2^2 + x_3^2 = n \\ & \frac{\varepsilon x_1^2}{b_1} - \frac{\varepsilon x_2^2}{b_2} + x_3 y_3 = m \\ & x_1^2 - x_2^2 + \frac{b_3}{a_1 - a_2} y_3^2 = \frac{l - a_3 n}{a_1 - a_2} = const. = \Gamma_0 \end{aligned}\right\} . \tag{10}$$

Instead of one of these integrals we can take the integral

$$x_3(a_1 - a_2) - y_3 \varepsilon = const. = \Gamma_1{}^* \tag{11}$$

which follow at once from the differential equations (9).

Then

$$\frac{dx_3}{dt} = \varepsilon \sqrt{(n - x_3^2)^2 - \left(\Gamma_0 - \frac{b_3}{a_1 - a_2} y_3^2\right)^2},$$

and with (11) the problem is reduced to elliptic functions.

The case D can be investigated in the same way as C.

(II). As is probably already known, lately the Russian mathematicians, W. Stekloff and A. Liapounoff discovered two new integrable cases of motion of a solid in infinite liquid (under no force) which A. Clebsch did not discover because of an error in his calculations.

In these cases the problem may be also reduced to ultra elliptic functions,

---

$$^* \Gamma_1 = (a_1 - a_2) \sqrt{n + \Gamma_0 \frac{b_2 + b_1}{b_2 - b_1} - 2m \frac{\varepsilon}{a_1 - a_2}}.$$

but in particular cases to simple elliptic functions. In Stekloff's case the kinetic energy is expressed by the formula:

$$2T = \Sigma\, \tau^2 a_1 (a_2^2 + a_3^2)\, x_1^2 + 2\tau a_2 a_3 x_1 y_1 + a_1 y_1^2$$

and the differential equations appear as follows:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= a_3 x_2 y_3 - a_2 x_3 y_2 - x_2 x_3 \tau a_1 (a_3 - a_2)\\
\frac{dx_2}{dt} &= a_1 x_3 y_1 - a_3 x_1 y_3 - x_3 x_1 \tau a_2 (a_1 - a_3)\\
\frac{dx_3}{dt} &= a_2 x_1 y_2 - a_1 x_2 y_1 - x_1 x_2 \tau a_3 (a_2 - a_1)\\
\frac{dy_1}{dt} &= y_2 y_3 (a_3 - a_2) - \tau a_1 (a_3 - a_2)(y_3 x_2 + x_3 y_2) +\\
&\qquad + \tau^2 (a_2 - a_3)(a_2 a_3 - a_1^2)\, x_2 x_3\\
\frac{dy_2}{dt} &= y_3 y_1 (a_1 - a_3) - \tau a_2 (a_1 - a_3)(y_1 x_3 + x_1 y_3) +\\
&\qquad + \tau^2 (a_3 - a_1)(a_3 a_1 - a_2^2)\, x_3 x_1\\
\frac{dy_3}{dt} &= y_1 y_2 (a_2 - a_1) - \tau a_3 (a_2 - a_1)(y_2 x_1 + x_2 y_1) +\\
&\qquad + \tau^2 (a_1 - a_2)(a_1 a_2 - a_3^2)\, x_1 x_2
\end{aligned}\right\} \qquad (12)$$

and admit a 4th algebraical integral of W. Stekloff:

$$\Sigma\, \tau^2 a_1 (a_1 - 2a_2 - 2a_3)\, x_1^2 + 2\tau a_1 x_1 y_1 - y_1^2 = const.$$

These differential equations admit a particular solution

$$y_1 = \tau (a_2 - a_3)\, x_1 \qquad y_2 = \tau (a_1 - a_3)\, x_2{}^{*} \qquad (13)$$

since the equations (1), (2), (4), (5) of the system (12) can be reduced by (13) to two differential equations

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= a_3 x_2 y_3 + \tau a_3 (a_2 - a_1)\, x_2 x_3\\
\frac{dx_2}{dt} &= -a_3 x_1 y_3 + \tau a_3 (a_2 - a_1)\, x_1 x_3\\
&\text{and the equations (3) and (6) give}\\
\frac{dx_3}{dt} &= 2\tau x_1 x_2 a_3 (a_1 - a_2)\\
\frac{dy_3}{dt} &= 2\tau^2 (a_2 - a_1)\, a_3 (2a_3 - a_1 - a_2)\, x_1 x_2
\end{aligned}\right\}. \qquad (14)$$

---

*In the same manner can be treated the solution: $y_2 = \tau (a_3 - a_1)\, x_2$ $y_3 = \tau (a_2 - a_1)\, x_3$ and the solution $y_3 = \tau (a_1 - a_2)\, x_3$ $y_1 = \tau (a_3 - a_2)\, x_1$.

The system (14) admits the integrals:

$$x_1^2 + x_2^2 + x_3^2 = n$$

$$x_1^2 - x_3^2 = \frac{1}{\tau^2 (a_2 - a_1)(2a_3 - a_1 - a_2)} y_3^2 + \Gamma$$

$$x_3 \tau (2a_3 - a_1 - a_2) + y_3 = \Gamma_1 \tag{15}$$

where n, Γ, Γ_1 are the constants of integration and we obtain:

$$\frac{dx_3}{dt} = \tau a_3 (a_1 - a_2) \sqrt{(n - x_3^2)^2 - \left(\Gamma + \frac{1}{\tau^2 (a_2 - a_1)(2a_3 - a_1 - a_2)} y_3^2\right)^2}$$

and by (15) the problem is reduced to elliptic functions. The motion in this case can be also resolved into two motions à la Poinsot.

In Liapounoff's case the kinetic energy is expressed by the formula:

$$2T = \Sigma b y_1^2 + 2c_1 x_1 y_1 + \frac{(c_2 - c_3)^2}{b} x_1^2,$$

and we obtain the differential equations of motion in the form:

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= b(x_2 y_3 - y_2 x_3) + (c_3 - c_2) x_2 x_3 \\
\frac{dx_2}{dt} &= b(x_3 y_1 - y_3 x_1) + (c_1 - c_3) x_3 x_1 \\
\frac{dx_3}{dt} &= b(x_1 y_2 - y_1 x_2) + (c_2 - c_1) x_1 x_2 \\
\frac{dy_1}{dt} &= (c_3 - c_2)\left\{x_3 y_2 + x_2 y_3 + \frac{x_2 x_3}{b}(2c_1 - c_2 - c_3)\right\} \\
\frac{dy_2}{dt} &= (c_1 - c_3)\left\{x_1 y_3 + x_3 y_1 + \frac{x_2 x_1}{b}(2c_2 - c_3 - c_1)\right\} \\
\frac{dy_3}{dt} &= (c_2 - c_1)\left\{x_2 y_1 + x_1 y_2 + \frac{x_1 x_2}{b}(2c_3 - c_1 - c_2)\right\}
\end{aligned}\right\} \tag{16}$$

and admit besides the three known integrals (2) the integral of A. Liapounoff:

$$\Sigma c_1((c_2 + c_3)\, x_1 + b y_1)^2 = const.$$

The differential equations (16) admit a particular solution:

$$y_1 = \frac{c_3 - c_2}{b} x_1 \qquad y_2 = \frac{c_3 - c_1}{b} x_2 .^* \tag{17}$$

As before, in Stekloff's case, we reduce the system (16) by (17) to four equations:

$$\frac{dx_1}{dt} = bx_2y_3 - (c_2 - c_1)x_2x_3$$

$$\frac{dx_2}{dt} = -(c_2 - c_1)x_1x_3 - bx_1y_3$$

$$\frac{dx_3}{dt} = 2(c_2 - c_1)x_1x_2$$

$$\frac{dy_3}{dt} = 2(c_2 - c_1)\frac{2c_3 - c_1 - c_2}{b} x_1x_2 .$$

The integrals of this system of differential equations we obtain in the form

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= n \\ x_1^2 - x_2^2 + \frac{b^2}{(c_1 - c_2)(2c_3 - c_1 - c_2)} y_3^2 &= \Gamma \\ y_3 + \frac{c_2 + c_1 - 2c_3}{b} x_3 &= \Gamma_1 \end{aligned} \tag{18}$$

and we find

$$\frac{dx_2}{dt} = (c_2 - c_1)\sqrt{(n - x_3^2)^2 - \left(\Gamma - \frac{b_3}{(c_1 - c_2)(2c_3 - c_1 - c_2)} y\right)^2}.$$

By (18) the problem is, as before, reduced to elliptic functions. The motion in this case can also be resolved into two motions à la Poinsot. These solutions are connected with some pseudo-elliptic integrals.

For instance let us consider the solution A of the case I. We have:

$$y_1b_1 = \varepsilon x_2 \quad y_2b_2 = \varepsilon x_1 \quad y_3 = \mathit{const.} = \quad \omega x_1^2 + x_2^2 + x_3^2 = n$$

$$\left(\frac{dx_3}{dt}\right)^2 = \varepsilon^2(n - x_3^2)^2 - 4b_3^2(m - \omega x_3)^2.$$

---

* The solution $y_2 = \frac{c_1 - c_3}{b} x_2$, $y_3 = \frac{c_1 - c_2}{b} x_3$ and the solution $y_3 = \frac{c_2 - c_1}{b} x_3$, $y_1 = \frac{c_2 - c_3}{b} x_1$ can be treated in the same manner.

Let us suppose that the axis Z fixed in space coincides with the axis of impulse and the position of the axes of x, y, z fixed in the body is given by the three Euler's angles θ, ϕ, ψ. Let p, q, r be the components of angular velocity in the directions of the axes of x, y, z. Then

$$\sin^2\theta\frac{d\psi}{dt}=\frac{1}{\sqrt{n}}(px_1+qx_2)$$

$$\frac{d\phi}{dt}=r-\cos\theta\frac{d\psi}{dt}.$$

But

$$p=\frac{\partial T}{\partial y_1}=b_1y_1=\varepsilon x_2,\quad q=\frac{\partial T}{\partial y_2}=b_2y_2=\varepsilon x_1,\quad r=\frac{\partial T}{\partial y_3}=b_3\omega$$

therefore

$$\frac{d\phi}{dt}=b_3\omega-\frac{2\varepsilon x_1x_2}{n-x_3^2}=b_3\frac{\omega n-2x_3m+\omega x_3^2}{n-x_3^2}.$$

But

$$\frac{1}{\sqrt{n}}(x_1+ix_2)=-\sin\theta\, e^{-i\phi}$$

so that

$$\phi=-\frac{1}{i}\,lg\,\frac{x_1+ix}{\sqrt{n_2-x_3^2}}.$$

We can conclude from this that the integral

$$\int\frac{\omega n-2x_3m+\omega x_3^2}{n-x_3^2}\,\frac{dx_3}{\sqrt{\varepsilon^2(n-x_3^2)^2-4b_3^2(m-\omega x_3)^2}}$$

is pseudo elliptic and

$$=-\frac{1}{b_3i}\,lg\,\frac{x_1+ix_2}{\sqrt{n-x_3^2}}=$$

$$=-\frac{1}{2ib_3}\,lg\,\frac{\frac{1}{\varepsilon}\sqrt{\varepsilon^2(n-x_3^2)^2-4b_3^2(m-\omega x_3)^2}+\frac{2ib_3}{\varepsilon}(m-\omega x_3)}{\sqrt{n-x_3^2}}.$$

This result can be verified by differentiating; and similar pseudo-elliptic integrals can be written also in the other cases.

Regarding investigations of Mr. Roger Liouville* I very much doubt that he really found a new integrable case of motion of a solid in infinite liquid as in the case of the problem of rotation of a heavy body about a fixed point, where he professed to have found a new integrable case. On the strength of which statement this case was entered under his name in different text books.*

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* Comptes rendus, 1896, II sém.

In 1898 my attempts to find integrals of the 8th and 12th power according to his method failed completely, as did also the attempts of Prof. S. Tshapliguine in Moscow. Whenever I succeeded in getting an integral according to Liouville's method it always turned out to be simply a reduction of already known integrals and only gave a new integral in the case of Mme. Sophie Kovalevsky. In the just published (January, 1906) article by Husson in the "Annales de Toulouse" this theorem is generally proved and so the integrable case of R. Liouville of rotation of a heavy body about a fixed point no longer exists.

JURJEW (Dorpat), Russia.

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* Appell. Traité de mécanique rationelle, t II.

On the Arrangement of the Real Branches of Plane Algebraic Curves.

By V. Ragsdale.

Introduction.

In the consideration of any problem relating to the number and arrangement of the real branches of plane algebraic curves, the division of circuits into the two classes *odd* and *even* is of fundamental importance.[*] An odd circuit can be met by a straight line in an odd number of points only; an even circuit in an even number of points. Two odd circuits have an odd number of intersections; an even and an odd circuit, or two even circuits have an even number of points in common. Hence, as Zeuthen shows (*Sur les differentes formes des courbes planes du quatrième ordre*, Math. Ann. VII, 1873, pp. 410-432), a non-singular curve of even order must be composed entirely of even circuits, and a non-singular curve of odd order must have one circuit odd and the rest even.[†]

In a paper published in 1876 (*Ueber die Vieltheiligkeit der ebenen algebraischen Curven*, Math. Ann. X, pp. 189-198) Harnack proved that a curve cannot have more than $p+1$ circuits, where p denotes the genus of the curve; also that for every value of p, a curve of some order does exist having $p+1$ real branches. In particular, if p be of the form $\frac{1}{2}(n-1)(n-2)$, there exists a non-singular n^{ic} with $\frac{1}{2}(n-1)(n-2)+1$ real branches. Later, Hilbert (*Ueber*

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[*] Von Staudt, *Geometrie der Lage*, 1847, p. 80.

[†] Zeuthen (*loc. cit.* p. 426) proves the existence of a quartic circuit with two double points, which is met by every straight line in at least two real points, and hence can not be projected into the finite part of the plane. Cayley (*On Quartic Curves*, Collected Papers, V. op. 361, 1865) points out that the sextic may be composed of one non-singular circuit which is met by every straight line in at least two real points. And C. A. Scott (*On the Circuits of Plane Curves*, Transactions of the American Mathematical Society, 1902) establishes the general theorem as to the existence of circuits that cannot be projected into the finite part of the plane. In the following pages, however, the only circuits that present themselves are those which can be projected into the finite, and for these the term *oval* is here employed.

die reellen Züge algebraischer Curven, Math. Ann. XXXVIII, 1890, pp. 115-138) considered certain possibilities of arrangement for the circuits of a non-singular n^{ic} when the maximum number of branches is present, and Hulburt (*A Class of New Theorems on the Number and Arrangement of the Real Branches of Plane Algebraic Curves,* American Journal of Mathematics, XIV, 1892, pp. 246-250) extended Hilbert's theorems to certain cases of curves with double points. Hilbert proved that for n even, not more than $\frac{1}{2}(n-2)$ of the $p+1$ ovals can be nested; that is, so situated that the first lies *inside*[*] a second, the second inside a third, and so on; and that curves of even order do exist having $p+1$ ovals, $\frac{1}{2}(n-2)$ of which are nested; similarly, that for n odd, not more than $\frac{1}{2}(n-3)$ ovals can be nested, if the maximum number of circuits is present, and that curves of odd order do exist having $p+1$ circuits, $\frac{1}{2}(n-3)$ of which are nested ovals.

A footnote to this paper[†] contains the statement that if the non-singular sextic have its maximum number of branches, eleven, these cannot all lie external to one another. Hilbert speaks of the process by which he arrived at this conclusion as "ausserordentlich umständlich," but no hint as to the character of the argument is given, and no proof of the statement has ever been published. However, if such a limitation on the arrangement of the ovals does exist for the 6^{ic}, there arises at once the question as to the existence of a similar limitation for all non-singular curves with the maximum number of branches. For curves of odd order no such restriction holds,—at least, in the form stated by Hilbert,—for it can be shown that a non-singular curve of odd order may have the maximum number of circuits with every oval lying outside the others. For the discussion of the question for curves of even order, however, it is convenient to cast Hilbert's statement into a slightly different form.

For the two types of the 6^{ic} given by Hilbert, the only types that can be derived by his method of generation, the arrangement of the ovals is the following:

(1) An oval O; 1 oval inside O, and 9 outside.

(2) An oval O; 9 ovals inside O, and 1 outside.

It is seen that the numbers of ovals "inside" and "outside" are interchanged in the two cases, and the natural inference is, that the law of arrangement to which the ovals are subject, is independent of the distinction between the "inside" and "outside" of a closed circuit, as defined by Von Staudt and Zeuthen, and, in fact, is based on no distinctive or permanent property of any one region of the plane. Thus the division of the plane by the curve $u=0$ into regions where u is positive and regions where u is negative offers a more promising basis for investigation of the problem, because of the element of arbitrariness introduced in ascribing to a certain region the positive rather than the negative sign. Suppose that the curve is non-singular and of even order, and that all its ovals have been projected into the finite. According to the usual convention let the sign be determined so that the expression u is positive at infinity. A region where u is negative may be a region bounded by a single circuit as in Fig. 1, or a region bounded by two or more circuits as in Fig. 2.

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[*] For the definition of "inside" and "outside" of a closed circuit see Von Staudt (*l. c.* p. 90) and Zeuthen (*l. c.* p. 410).

[†] l. c. pp. 118-119.

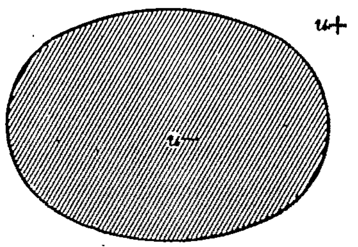

FIG. 1.

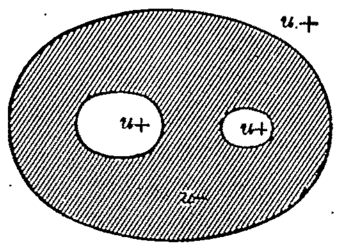

FIG. 2.

Each additional boundary introduces a new positive region. If such a boundary, or an oval which cuts off in the midst of a region where u is negative a region in which u is positive, be called an *internal* oval, and an oval which cuts off in the midst of a region where u is positive a region in which u is negative, an *external* oval, Hilbert's statement can be expressed as follows: *If the non-singular sextic have its maximum number of branches, at least one of the eleven ovals must be internal*;—that is, not more than ten of the eleven ovals can be external. There is as yet no formal proof forthcoming for this statement in either its original or altered form, but as curves of higher order are investigated a most interesting law governing the arrangement of the ovals presents itself so persistently, and in curves of such widely different types, as to give strong reasons for belief in the existence of a general theorem. It is found that the 8^{ic}, 10^{ic}, 12^{ic}, 14^{ic}, ..., with the maximum number of circuits, will have respectively 3, 6, 10, 15, ..., or more internal ovals. And in general, *if the non-singular* $2n^{ic}$ *have the maximum number of branches, at least* $\frac{1}{2}(n-1)(n-2)$ *of the* $p+1$ *ovals must be internal*; *or not more than* $n^2+\frac{1}{2}(n-1)(n-2)$ *can be external.*

As will be shown later, the only processes by which curves with the maximum number of branches have been derived, yield curves of even order whose circuits conform to this law of arrangement. These are the two processes employed by Harnack and Hilbert. The *Harnack process* offers two modes of generation, each of which determines a distinct law of arrangement for the circuits of the derived curve C_{2n}; but these two laws and all modifications of them which arise from combinations of the two modes of generation differ only in the distribution of the internal ovals. The number in every case is $\frac{1}{2}(n-1)(n-2)$. For example, of the 22 ovals of the 8^{ic}, 3 are internal, though these 3 may be distributed in two ways (Fig. $3_{(b),(c)}$). Of the 37 ovals of the 10^{ic}, 6 are internal, though these 6 may be distributed in four ways (Fig. $3_{(d),(e),(f),(g)}$).

The *Hilbert process* gives less simple arrangements of the circuits. Hilbert's own statement is, that if the $2n^{ic}$ have the maximum number of nested ovals, $n-1$, the remaining ovals must be external to one another and may be distributed in various ways in the annular regions bounded by two successive nested ovals, and in the region lying outside the nest. It is shown (p. 389) that the simplest arrangement of these remaining ovals is represented by the following scheme, which gives the number of ovals in the annular regions, beginning with the innermost ring, $0, 2, 4, 6, 8, \ldots 2n-10, 2n-8, 2n-6$; the other ovals lie outside the nest entirely. In this case the number of internal ovals is exactly $\frac{1}{2}(n-1)(n-2)$. But for all curves of order $2n\,(2n>6)$, the process gives choice of three distinct modes of generation, and hence affords various possibilities for the arrangement of the circuits. It is still true, however, that no type of $2n^{ic}$ obtained has less than $\frac{1}{2}(n-1)(n-2)$ internal ovals.

Both these processes are based upon the principle of small variation from a special degenerate curve. This reducible curve is composed of an $m-k^{ic}$ with the maximum number of circuits and an auxiliary curve of order k which bears a certain specified relation to the $m-k^{ic}$. The m^{ic} obtained has the maximum number of circuits, and bears a relation to the auxiliary curve similar to that possessed by the $m-k^{ic}$. The two methods differ only in the type of auxiliary curve employed. The Harnack process is characterized by the use of the straight line as auxiliary curve; the Hilbert process by the use of the ellipse. Hulburt has proved[*] that in the generation of curves by the method of small variation, the only auxiliary curves that will yield the maximum number of branches on the derived m^{ic}, are the straight line and the conic; that is, the only processes of this type that will give curves with the maximum number of circuits are the Harnack and Hilbert methods, and modifications of the Hilbert method due to the use of other conics as the auxilary curve. Hence if all non-singular curves with the maximum number of branches are obtainable by the method of small variation, the law which expresses the arrangement of ovals for curves derived by the Harnack and Hilbert processes becomes a general law, and holds for *all* curves of even order with the maximum number of circuits. But whether the law is of perfect generality or not, it is of interest to investigate more fully the various types of curves that can be derived by these different methods.

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[*] l. c. p. 250.

CURVES WITH THE MAXIMUM NUMBER OF BRANCHES DERIVED BY THE HARNACK AND HILBERT PROCESSES OF GENERATION.

Curves Derived by the Harnack Process.—Let C_{n-1} be a non-singular curve of order $n-1$ with the maximum number of circuits, and let a straight line v meet one circuit of the curve in $n-1$ real and distinct points which have the same order of succession on C_{n-1} as on v. Harnack shows then that by a proper choice of δ and the straight lines, $l_1, l_2, l_3, \ldots l_n$, $C_n \equiv v \,.\, C_{n-1} + \delta \,.\, \prod_{i=1}^{i=n} l_i = 0$ can be made to represent a curve of order n having properties corresponding to those of the $n-1^{ic}$. Certainly for $n=3$, such an $n-1^{ic}$ exists; viz., a conic cut in two real points by a straight line v. Let three lines l_1, l_2, l_3 be chosen so that they cut the infinite segment of v. Then the cubic represented by the equation $C_2 \,.\, v + \delta \,.\, l_1 l_2 l_3 = 0$ will pass through the intersections of C_2 and v with $l_1 l_2 l_3$, and for a small value of δ will have the maximum number of circuits, two. Moreover, the infinite branch is cut by the straight line v in three real points (Fig. 1, Plate I). The quartic can be derived from the cubic in the same way. Let the lines l_1, l_2, l_3, l_4 be so chosen that they cut the same segment of v; then for a proper choice of δ, the equation, $C_4 \equiv C_3 \,.\, v + \delta \,.\, \prod_{i=1}^{i=4} l_i = 0$, will represent a quartic with four ovals, one of which meets v in four real points (Fig. 2, Plate I). Similarly, the quintic with the maximum number of circuits can be obtained from the quartic, and the sextic from the quintic, and so on.

The restrictions imposed on the $n-1^{ic}$, viz.: (1) That the $n-1^{ic}$ must have the maximum number of branches, (2) that a straight line v must cut C_{n-}, in $n-1$ real and distinct points, (3) that all these points must lie on the same circuit of C_{n-1}, are necessary in order that C_n may have the maximum number of branches. Let that circuit of the $n-1^{ic}$ which is cut by v in $n-1$ real points be called the generating circuit g_{n-1}. Restrictions (2) and (3) require that g_{n-1} be the infinite branch, if $n-1$ is odd. There are also certain restrictions that must be imposed on the lines $l_1, l_2, l_3, \ldots l_n$. The points in which C_n cuts v are determined by the points common to v and the lines $l_1, l_2, l_3, \ldots l_n$. If an odd number of these lines cut any finite segment determined on v by C_{n-1} (*n odd*) or *any* segment (*n even*) the number of circuits of C_n falls short of the maximum number. It is clear that by admitting imaginary lines there can be obtained from the given $n-1^{ic}$, n^{ics} with the maximum number of branches, and cut by v in $0, 2, 4, \ldots$ or n real points if n is even, or in $1, 3, 5, \ldots n$ real points if n is odd; and also that these points of intersection, if more than two, may lie on different circuits. But for the generation of the $n+1^{ic}$ with the maximum number of circuits from the n^{ic} all the intersections of C_n and v must be real and lie on the same circuit g_n. Hence the straight lines $l_1, l_2, l_3, \ldots l_n$ must be chosen to cut the same segment of v, and in n real and distinct points. If n is odd, this segment must be the infinite segment; if n is even, the segment may be any one, finite or infinite. The general arrangement of the circuits of the n^{ic} with the maximum number of branches is the same whether the intersections of C_n and v are all real and lie on the same circuit or not. The difference in the two cases manifests itself in the number of branches on the $n+1^{ic}$ and curves of higher order derived from the n^{ic}. Hence, in considering the different types of n^{ics} with the maximum number of branches, it is necessary to take account only of those cases where the lines $l_1, l_2, \ldots l_n$ are subject to such restrictions as allow the process to be continued.

In the generation of the quintic from the quartic (Fig. 3, Plate 1), one circuit of C_4 must cut v in four points. Of the four segments of g_4, two with their corresponding segments of v give rise to two ovals lying external to one another. Of the other two, one, together with the infinite segment of v, generates the infinite branch of the 5^{ic}; the other, with its corresponding segment of v, produces an oval on the 5^{ic}, which must lie in one of the regions bounded by a segment of v and a segment of the infinite branch of C_5 (Fig. 3, Plate I). Hence the 6^{ic} arising from this 5^{ic} must have one oval lying inside another; the other ovals are external, five representing the five remaining ovals of the 5^{ic}, and four generated by the other segments of v and the infinite branch of C_5 (Fig. 4, Plate I).

Though with reference to the arrangement of the circuits there is only one kind of 6^{ic} obtained, there are with regard to the 6^{ic} two essentially different positions of the lines $l_1, l_2, \dots l_6$, which are of importance in the generation of curves of higher order. According as these lines cut that segment of v which, together with one segment of C_5, encloses a region containing an oval, or one of the other segments, the generating oval encloses (a) one other oval (Fig. 4, Plate I), or (b) includes no oval (Fig. 5, Plate I).

From the 6^{ic} of type (a) is derived a 7^{ic} with the maximum number of circuits each oval of which lies external to the others ; from the 6^{ic} of type (b), a 7^{ic} with the maximum number of branches and with two ovals nested. It is seen from Figs. 4, 5, Plate I, that of the six segments of g_6, three lie on one side of v and with segments of v give rise to three ovals on C_7 which are external to one another; but of the other three, two lie in the region bounded by the third and the finite segment $x_1 x_6$ of v. This third segment of g_6 and the infinite segment $x_6 x_1$ of v give rise to the infinite branch of the 7^{ic}. The other two segments, with their corresponding segments of v, generate two ovals external to one another, but situated in one of the seven regions formed by the intersections of g_7 and v. In this region must lie also the representative of the oval, if any, which is encircled by the generating oval. Hence the 7^{ic} of the first type must have three ovals lying in one of the 7 regions bounded by a segment of v and a segment of g_7, and the 7^{ic} of the second type (the one that has the pair of nested ovals) must have two ovals lying in one of these seven regions. In both cases the remaining six regions contain no ovals ; the arrangement of the other ovals is similar to the arrangement of those on the sextic from which they are derived. Hence *the* 8^{ic} *generated by the* 7^{ic} *of the first type will have one oval which encloses three others.* There is only one type of 8^{ic} obtained, but the generating oval may be that which encircles three others, or one which includes none. Thus, in passing to the 9^{ic}, there is a choice again between two modes of generation. *The* 8^{ic} *generated by the* 7^{ic} *of the second type has one oval enclosing two others and one including a single oval;* and two cases arise as before, according as the generating oval is the oval which encloses two others, or one which includes none. Hence, just as two types of the 7^{ic} were derived from the one form of the 6^{ic}, two types of the 9^{ic} are generated by each of the two forms of the 8^{ic}.

By exactly the same argument it can be shown that each type of the $2m - 2^{ic}$ gives rise to two types of the $2m - 1^{ic}$, determined by the character of the generating oval, g_{2m-2}, which may enclose a number of other ovals or may contain none at all. But since in the generation of the $2m^{ic}$ from the $2m - 1^{ic}$

the straight lines $l_1, l_3, \ldots l_{2m-1}$ must cut the infinite segment of v, each type of the $2m - 1^{ic}$ can give rise to only one type of the $2m^{ic}$. In this way two types of the $2m^{ic}$ arise from each form of the $2m - 2^{ic}$ $(2m > 6)$, and as the process is continued $(2m = 8, 10, \ldots 2n)$ there will arise 2^{n-3} types of the $2n^{ic}$. These, however, do not differ in the number of internal ovals. For with the exception of the ovals derived from v and the infinite branch of C_{2n-1}, the arrangement of the ovals of the $2n^{ic}$ is similar to that of the $2n - 1^{ic}$. The infinite branch of C_{2n-1} and v form $2n - 1$ regions, each bounded by a single segment of C_{2n-1} and a single segment of v, and in one of these regions must lie the representatives of all the ovals, if any, included by the generating oval of C_{2n-2}, as well as $\frac{1}{2}(2n - 2) - 1$ of the $2n - 3$ ovals which arise from segments of g_{2n-2} and v. No other of the $2n - 1$ regions contains an oval. Hence the oval on the $2n^{ic}$ derived from the segments of C_{2n-1} and of v which bound this region, will include $n - 2$ ovals and also those representing the ovals which were contained by g_{2n-2}. Thus whatever be the type of $2n^{ic}$, the number of its internal ovals exceeds by $n - 2$, the number on the $2n - 2^{ic}$.

For the 6^{ic} the number of internal ovals is 1,
for the 8^{ic}, $1 + 2$,
for the 10^{ic}, $1 + 2 + 3$,
for the 12^{ic}, $1 + 2 + 3 + 4$, etc.

Hence on *every* curve of even order $(2n)$ with the maximum number of circuits there are $1 + 2 + 3 + 4 + \ldots + n - 4 + n - 3 + n - 2$, i.e. $\frac{1}{2}(n-1)(n-2)$ internal ovals, and hence $n^2 + \frac{1}{2}(n - 1)(n - 2)$ external ovals.

Though the number of internal ovals is the same for all $2n^{ics}$ thus generated, the distribution differs from type to type. If on each $2m^{ic}$ $(2m = 2, 4, 6, \ldots 2n - 2)$ from which the $2n^{ic}$ is derived the generating oval be one which contains others, that is, if the first mode of generation be used throughout, all the internal ovals lie inside the same oval (Fig. $3_{(b), (d), (h)}$). If, however, the generating oval be one which encloses no other, then among the *additional* circuits formed in passing from a curve of order $2n - 2$ to a curve of order $2n$, there is just one oval which includes others, and this contains $n - 2$. Thus the $2n^{ic}$ which is derived by the second mode of generation throughout has its internal ovals distributed in $n - 2$ different ovals, in groups of $1, 2, 3, 4, \ldots n - 3, n - 2$ (Fig. $3_{(c), (g), (o)}$). Combinations of the two modes of generation afford $(2^{n-3} - 2)$ other arrangements of the internal ovals, but there are certain restrictions to which the distribution is subject. No set of $1, 2, 3, \ldots, n - 2$ ovals can be separated, and combinations can be made only of successive sets (Fig. $3_{(e), (f), (i), (j), (k), (l), (m), (n)}$).

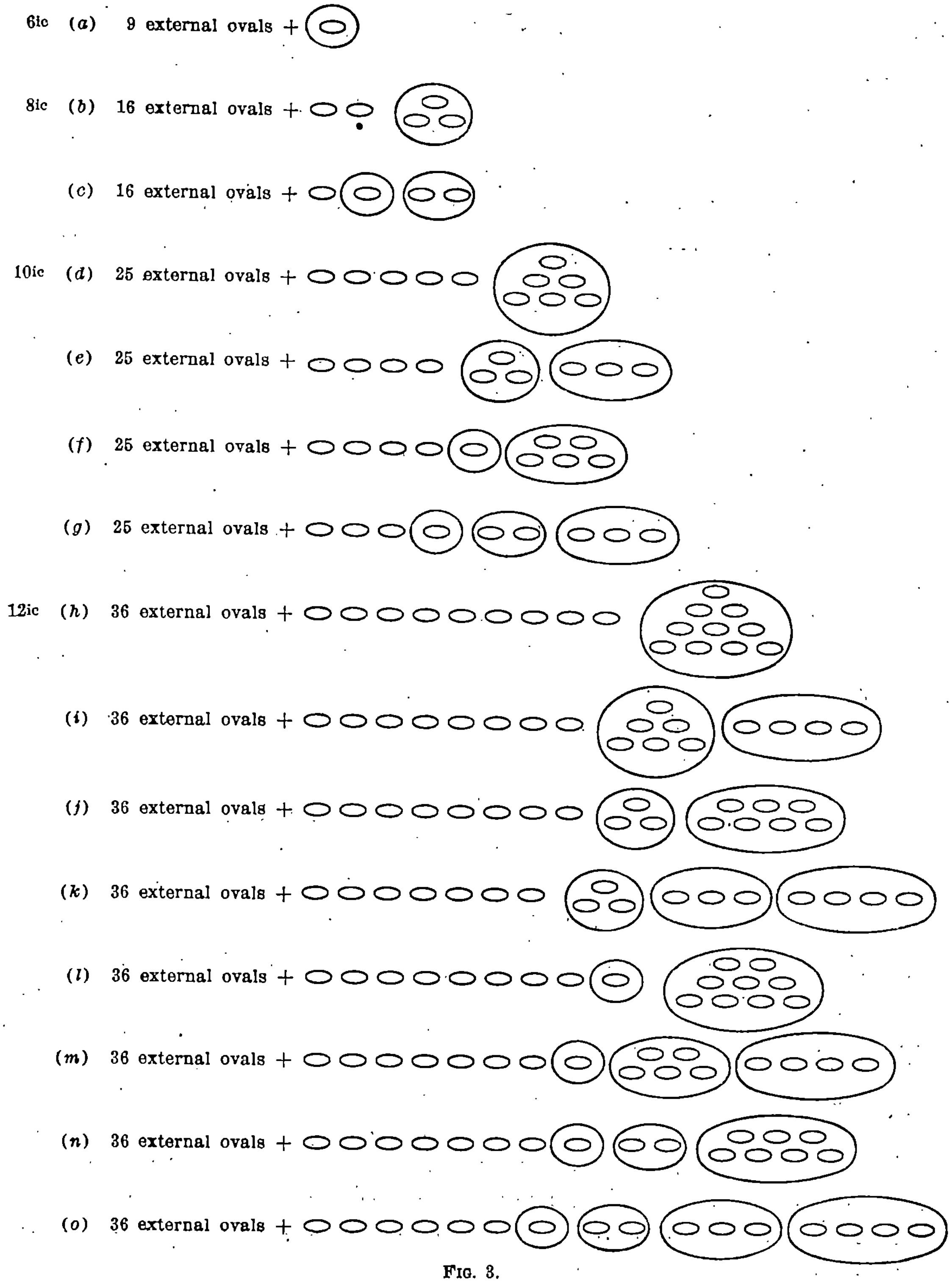
6ic (a) 9 external ovals +
8ic (b) 16 external ovals +
(c) 16 external ovals +
10ic (d) 25 external ovals +
(e) 25 external ovals +
(f) 25 external ovals +
(g) 25 external ovals +
12ic (h) 36 external ovals +
(i) 36 external ovals +
(j) 36 external ovals +
(k) 36 external ovals +
(l) 36 external ovals +
(m) 36 external ovals +
(n) 36 external ovals +
(o) 36 external ovals +

Fig. 3.

If the non-singular curve of even order has not the maximum number of circuits, the question arises, whether in this case the number of external ovals can exceed $n^2 + \frac{1}{2}(n-1)(n-2)$. The generating oval of C_{2n-2} with v gives rise to the infinite branch of C_{2n-1} and to $2n-3$ ovals, $n-2$ of which become internal ovals on the $2n^{ic}$. The remaining $n-1$ ovals, and all additional ovals arising from g_{2n-1} and v become external ovals. It is obvious that the $2n-2$ straight lines $l_1, l_2, l_3, \ldots l_{2n-2}$ could have been so chosen that the position of g_{2n-2} with regard to v would have been that indicated by Fig. 4, and hence that none of the $n-2$ internal ovals would have appeared on the $2n^{ic}$. This is the only way, however, in which the presence of internal ovals can be prevented without decreasing the number of external ovals, and this method admits of no increase in the number of the latter. Hence the conclusion can be drawn that no 6^{ic} derived by the Harnack process, can have more than 10 external ovals, no 8^{ic} more than 19 external ovals, no 10^{ic} more than $31, \ldots$, no $2n^{ic}$ more than $n^2 + \frac{1}{2}(n-1)(n-2)$, even though the number of circuits on the curve be less than the maximum number.

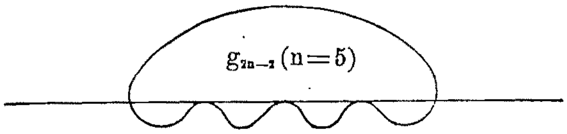

FIG. 4.

In support of the statement that on *curves of odd order*, the ovals may be so arranged that each lies outside the others, it was seen (p. 383) that from the 6^{ic} whose generating circuit included another oval, a 7^{ic} can be derived whose ovals lie external to one another. And in general from every $2n^{ic}$ whose generating oval includes *all* the internal ovals can be derived a $2n+1^{ic}$, all of whose ovals are external to one another. For as C_{2n+1} is generated from C_{2n}, the generating oval opens out, so to speak, to form with the infinite segment of v, the infinite branch of the $2n+1^{ic}$, thus leaving the ovals which it contained free of any encircling oval.

Curves of Even Order derived by the Hilbert Process.—For the types of $2n^{ics}$ which present themselves by the Harnack process, all the ovals which lie "inside" others satisfy the definition of "internal" ovals. Not so for curves with nested ovals; for within the annular regions bounded by two successive ovals of the nest, beginning with the outermost ring the expression, C_{2n}, is alternately negative and positive. The internal ovals, then, are to be looked for only in these negative regions, that is, in the 1st, 3rd, 5th, ... etc., from the outside. The circuits lying in the 2nd, 4th, 6th, ... regions lie "inside" certain ovals but are themselves external ovals.

It has already been mentioned that the Hilbert method of generating curves with the maximum number of circuits, differs from the Harnack process only in the use of the ellipse instead of the straight line as auxiliary curve. The process, as given by Hilbert, applies to curves of both odd and even orders, but here only curves of even order will be considered. Let C_{2n} be a curve of even order with the maximum number of circuits, $p+1$, and the maximum number of nested ovals, $n-1$; and assume that an ellipse, E_2, can be drawn to enclose one or more of the nested ovals and cut one of the non-nested ovals, g_{2n}, in $4n$ points which have the same order of succession upon C_{2n} as upon the ellipse. On a segment, S, of E_2, but not that which with a segment of g_{2n} encloses the one or more nested ovals inside the ellipse, let $4n+4$ points be chosen and through these points let $2n+2$ straight lines, $l_1, l_2, \dots l_{2n+2}$, be drawn, connecting the first point with the second, the third with the fourth, and so on.* Then for a small value and the proper sign of δ the equation $C_{2n} \,.\, E_2 + \delta \,.\, \prod\limits_{i=1}^{i=2n+2} l_i = 0$ represents a curve of order $2n+2$, which has the maximum number of branches, $p+1$, the maximum number of nested ovals, n, and satisfies all other conditions analogous to those assumed for the $2n^{ic}$.† Hence if a $2n^{ic}$ exists satisfying the assumed conditions, from it can be derived a $2n+2^{ic}$, subject to similar conditions, from this a $2n+4^{ic}$, and so on. For the case $2n=4$, such a curve does exist.

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* It can be seen, as in the preceding method, that the assumptions made for the $2n^{ic}$ are necessary for the maximum number of circuits, or for the maximum number of nested ovals on the $2n+2^{ic}$, or for the continuation of the process beyond the generation of C_{2n+2}. The assumption that the ellipse must enclose at least one of the nested ovals and the restriction made on the segment, S, are not given by Hilbert but are shown by Hulburt (*Topology of Algebraic Curves*, Bull. N. Y. Math. Soc. I, 1891-2, p. 197) to be necessary for the continuation of the process. Otherwise the curves of higher order would not have the maximum number of nested ovals. The restriction of the $4n+4$ points to the *same* segment is necessary in order that curves of order $> 2n+2$ may have the maximum number of circuits.

† With the exception of the oval, g_{2n}, each circuit of C_{2n} gives rise to a circuit of C_{2n+2}. Also the boundaries of the $4n$ regions formed by the intersections of g_{2n} and E_2 generate $4n$ ovals. Hence *the total number of branches* $= \frac{1}{2}(2n-1)(2n-2) + 4n = p+1$. The ovals arising from the nested ovals are nested, and one of the $4n$ ovals generated by the segments of g_{2n} and E is itself a nested oval. Hence the number of nested ovals $= \frac{1}{2}(2n-2) + 1 = \frac{1}{2}(\overline{2n+2}-2)$. Moreover the ellipse encloses one or more of the nested ovals, and a non-nested oval, g_{2n+2}, cuts E_2 in $4n+8$ points whose order of succession is the same on the oval and the ellipse.

Let C_2 and E_2 represent two ellipses cutting each other in four real points. On a segment, S, of E_2 let 8 points be chosen and joined by the straight lines l_1, l_2, l_3, l_4, the 1st point from one end of the segment with the 2nd, the 3rd with the 4th, and so on. Then the equation $C_n \,.\, E_2 + \delta \,.\, \prod_{i=1}^{i=4} l_i = 0$, for a proper choice of δ, will represent a quartic satisfying the assumed conditions (Figs. 1, 2, Plate II). And therefore for all order values, $2n$, curves do exist satisfying similar conditions.

There is only one type of quartic obtained, but two cases arise from the two possible positions of the lines l_1, l_2, l_3, l_4 with reference to the auxiliary ellipse. If in the derivation of the 4^{ic} from the conic, C_2, no real points had been chosen on a segment, S, of E_2, the quartic would have consisted of two ovals inside E_2 and two ovals outside. Hence according as the 8 points chosen lie on a segment, S, outside C_2 or on a segment inside C_2, one of the two ovals inside E_2, or one of the two ovals outside E_2, becomes the generating oval. Each of these two quartics gives rise to a distinct type of 6^{ic} $(C_4 \,.\, E_2 + \delta \,.\, \prod_{i=1}^{i=6} l_i = 0)$ with the required properties (Figs. 3, 4, Plate II). It is easily seen that in the generation of each type of 6^{ic}, there are possible three essentially different positions of the 12 points which, taken in pairs, determine the 6 straight lines $l_1, l_2, \ldots l_6$. For one position, the ellipse, E_2, is cut by a non-nested oval of C_6 which would otherwise lie inside the ellipse; for another, by a non-nested oval which would otherwise lie outside the ellipse; for the third, by a nested oval. And, in general, the same possibilities arise in the derivation of the $2n^{ic}$ from the $2n - 2^{ic}$, thus affording three modes of generation of the $2n + 2^{ic}$ from a given $2n^{ic}$. The third case, however, as Hulburt points out, leads to curves with less than the maximum number of nested ovals. It will be found later that after an application of the third mode of generation, a fourth mode becomes possible. All four modes yield curves with the maximum number of circuits, but only the first and second admit also the maximum number of nested ovals.

If at each stage of the generation of the $2n^{ic}$ from the curves of lower order, one of the non-nested ovals inside the ellipse be taken as the generating oval, that is, if a *curve be derived* by *the first mode of generation* throughout, the circuits situated in the $n - 2$ annular regions determined by the nested ovals are distributed according to a perfectly regular scheme. The non-nested oval of the $2n - 4^{ic}$ which cuts the ellipse forms with the latter, $4n - 8$ regions in which are generated $4n-8$ new ovals of the $2n-2^{ic}$. Of the $2n-4$ of these which lie inside the ellipse, one is a nested oval; another is to be taken as the generating oval; hence in the region bounded by the outer oval of the nest and those segments of the ellipse and the generating oval which give rise to the *new* nested oval of C_{2n}, there are situated exactly $2n-6$ ovals. Therefore the $2n^{ic}$ will have in the last annular region formed $2n-6$ ovals. Since one new annular region is formed at each stage of the generation, and the arrangement of the ovals lying in this region is not disturbed as curves of higher order are generated, there corresponds to each curve, C_{2m}, ($2m \geqq 6$) from which the $2n^{ic}$ is derived, one particular ring. For the 6^{ic}, the number of ovals in the ring between the two nested ovals $=0$; for the 8^{ic}, the number of ovals in the 1st, or innermost, and the 2nd rings $=0,\ 2$; for the 10^{ic} the number of ovals in the 1st, 2nd, and 3rd rings $=0,\ 2,\ 4$; and so on; for the $2n^{ic}$ the number of ovals in the 1st, 2nd, 3rd, ... $n-2^{th}$ rings $=0,\ 2,\ 4,\ 6, \ldots .\ 2n-10,\ 2n-8,\ 2n-6$. But in the consideration of the number of internal or external ovals, that nested oval which forms the inner boundary of a ring itself, belongs to the group of ovals in that region; hence the foregoing scheme becomes

$$1,\ 3,\ 5,\ 7, \ldots\ldots\ 2n-9,\ 2n-7,\ 2n-5,$$

and these groups are alternately internal and external ovals, or vice versa. Therefore for $n-2$ even (Fig. 5, Plate II, $n=4$),

$$\text{the number of internal ovals} = 2n-5+2n-9+\ldots+7+3 = \tfrac{1}{2}(n-1)(n-2);$$

for $n-2$ odd,

$$\text{the number of internal ovals} = 2n-5+2n-9+\ldots+5+1 = \tfrac{1}{2}(n-1)(n-2).$$

If in the generation of the $2n^{ic}$ one of the non-nested ovals lying outside the ellipse be taken at each stage as the generating oval, that is, if the *curve be derived by the 2nd mode of generation throughout*, there is obtained a similar arrangement of circuits in the annular regions. Beginning with 3, however, the series is reversed ($1,\ 2n-5,\ 2n-7,\ 2n-9, \ldots .\ 7,\ 5,\ 3$), since by this process after the generation of the 6^{ic}, the nest is built up from the outside inward; and the innermost ring contains not only one oval in accordance with the scheme, but also all, save one, which by the other process lay in that part of the plane exterior to the nest. Therefore for $n-2$ *even* when within the innermost ring, the expression, C_{2n}, is positive (Fig. 6, Plate II), the number of internal ovals $= \frac{1}{2}(n-1)(n-2)$, but for $n-2$ *odd* the number of internal ovals

$$= n^2 + \tfrac{1}{2}(n-1)(n-2) - 1 > \tfrac{1}{2}(n-1)(n-2).$$

Combinations of the 1st and 2nd modes of generations give other types of curves. At each stage of the development of the curve, C_{2n}, from the conic, C_2, either mode of derivation may be adopted; hence from the quartic of which there is only one type, are derived two 6^{ics}, from each 6^{ic}, two 8^{ics}, and so on; the number increasing in geometrical ratio as n increases. Therefore for the $2n^{ic}$, if the two regular types just discussed be included, the number of types of curves with the maximum number of circuits and the maximum number of nested ovals, is 2^{n-2}. By a combination of the two modes, the nest is built up alternately from the inside outward and from the outside inward. If the curve, C_{2m-2}, is derived throughout the process by the first mode of generation, all of its nested ovals lie inside the ellipse, E_2. For the generation of the curve, C_{2m}, from this, let a change be made to the 2nd mode; then the *new* nested oval of C_{2m} lies outside the ellipse, and the annular region which corresponds to the curve, C_{2m}, is the one in which the ellipse is situated. As curves of higher order are generated all the *new* ovals appear in this region, and from it are cut off successively the *new* annular regions lying inside or outside the ellipse according as they are formed by the 1st or 2nd mode of generation. The curve is built up in such a manner that there are formed sets of annular regions, (1) a inside the ellipse, (2) a' outside the ellipse, (3) b inside the ellipse, (4) b' outside the ellipse, and so on. Except in those rings which are formed at stages where a change in the process occurs, the number of ovals in each annular region is the same as that in the corresponding region when one type of generation is used throughout,—that is in the last annular region formed whether outside or inside the ellipse there are $2n-6$ ovals. But consider the ring formed in the derivation of C_{2m} from C_{2m-2} where a change is made from the 1st to the 2nd mode of generation. The outer nested oval of C_{2m-2} is one of the $2m-4$ ovals arising from segments of g_{2m-4} and E_2, and lying inside the ellipse; and the generating oval of C_{2m-2} is one of the $2m-4$ ovals outside the ellipse. Thus $2m-5$ ovals are left in the region between the ellipse and the outer nested oval of C_{2m-2}. Hence as C_{2m} is derived from this $2m-2^{ic}$, an annular region is formed which contains $2m-5+2m-2$ ovals inside the ellipse and $2m-3$ lying outside. If the second process is continued through the generation of C_{2p-2}, the number of ovals in each new annular region formed follows the regular scheme up to this stage, and the appearance of every group of $2q - 6$ ovals in a ring outside the ellipse $(2q = 2m + 2, 2m + 4, \ldots . 2p - 2)$, is accompanied by the appearance of a group of $2q - 4$ ovals in the region containing E_2 and inside the ellipse. Hence this ring contains

$2m - 5 + \sum_{m+1}^{p-1}{}_{q}\, 2q - 4 + 2p - 4$ ovals inside the ellipse and $2p - 5$ outside.

If at this stage a change is made back to the first type of generation, one of the $2p - 4$ ovals inside the ellipse must be taken as the generating oval, g_{2p-2}, and hence in the new annular region formed for C_{2p}, a region which lies inside the ellipse, there are $2m - 5 + \sum_{m+1}^{p-1}{}_{q}\, 2q - 4 + 2p - 5$ ovals, and in the region which contains the ellipse $2p - 3$ ovals inside the ellipse and $2p - 2 + 2p - 5$ outside. Hence as C_{2n} is derived, if there is a change from the 1st mode of generation to the 2nd in the derivation of C_{2m}, from the 2nd to the 1st in the derivation of C_{2p}, from the 1st to the 2nd in the derivation of C_{2s}, and so on, the scheme of arrangement of the ovals in the annular regions beginning with the innermost ring is represented by the following sets of groups, $a, a', b, b', c, c', \ldots .$,

$$(a)\,|\,0,\ 2,\ 4,\ 6,\ \underrightarrow{8, \ldots . 2m - 8\,|,\ (b)\,|\,(2m} - 5 + \underrightarrow{\sum_{m+1}^{p-1}{}_{q}\, 2q - 4 + 2p - 5)},$$
$$2p - 4,\ 2p - 2, \ldots . 2s - 8\,|,$$

$$(c)\,|\,(2s - 5 + \sum_{s+1}^{t-1}{}_{q}\, 2q - 4 + 2t - 5),\ 2t - 4,\ 2t - 2, \ldots . 2n - 6,$$
$$(2n - 2 + 2n - 3 + \sum_{t+1}^{n}{}_{q}\, 2q - 4 + 2t - 5)\,|,$$

$$(b')\,|\,2t - 8, \ldots . \underleftarrow{2s - 2,\ 2s - 4,\ (2s - 5 + \sum_{p+1}}^{s-1}{}_{q}\, 2q - 4 + 2p - 5)\,|,$$
$$(a')\,|\,2p - 8, \ldots . \underleftarrow{2m - 2},\ 2m - 4\,|;$$

or if in the enumeration of the ovals in the annular regions the nested oval which forms the inner boundary of the ring be included, the foregoing scheme becomes

$$
\begin{aligned}
&(a)\,|\,1,\ 3,\ 5,\ 7, \ldots\, 2m-7\,|,\ (b)\,|\,(2m-5+\sum_{m+1}^{p-1}{}_{q}\, 2q-4+2p-4), \\
&\qquad\qquad\qquad\qquad\qquad\qquad 2p-3,\ 2p-1, \ldots\, 2s-7\,|, \\
&(c)\,|\,(2s-5+\sum_{s+1}^{t-1}{}_{q}\, 2q-4+2t-4),\ 2t-3,\ 2t-1, \ldots\, 2n-5, \\
&\qquad\qquad\qquad\qquad (2n-1+2n-3+\sum_{t+1}^{n}{}_{q}\, 2q-4+2t-5)\,|, \\
&(b')\,|\,2t-7, \ldots\, 2s-1,\ 2s-3,\ (2s-4+\sum_{p+1}^{s-1}{}_{q}\, 2q-4+2p-5)\,|, \\
&\qquad\qquad\qquad\qquad\qquad (a')\,|\,2p-7, \ldots\, 2m-1,\ 2m-3\,|.
\end{aligned}
$$

If the 2nd mode of generation is applied first, then the region which includes the ellipse corresponds to the 6^{ic}. The foregoing scheme applies, but $2m$ has the value 6, and group (a) disappears.

If the annular regions be considered in the order of their formation but that region in which the ellipse is situated be left to the last, the series of numbers representing the groups of ovals is $(a)\,|\,1,\ 3,\ 5,\ 7, \ldots\, 2m-7\,|$, $(a')\,|\,2m-3,\ 2m-1 \ldots\, 2p-7\,|$, etc. Comparison with the regular series obtained when the first mode of generation is employed throughout shows that after $2m-7$, each number in this series is greater by 2 or by a quantity exceeding 2, than the corresponding number in the regular series. If in the last region of each set, a, a', b, b', ... the expression, C_{2n}, is negative, or more briefly if the last region in each set is negative except in that set in which the ellipse is situated, the series which represents the number of *internal* ovals is

$$
\begin{aligned}
&\underset{\text{or } 3,\ 7,\ 11,}{1,\ 5,\ 9,} \ldots\, 2m-11,\ 2m-7,\ 2m-3,\ 2m+1, \ldots\, 2p-7,\ 2p-3, \ldots\ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2s-7,\ 2s-3, \ldots\ldots
\end{aligned}
$$

in exact agreement with the regular series representing the number of internal ovals when the 1st mode of generation is used throughout. For though the first region of (a') is negative the number of its ovals exceeds by 4 the number in the last region of (a), and as the first regions of all the other sets b, b', c, c', ... are necessarily positive there can be no break in the series as passage is made from set to set. But when there is a change made from one mode of generation to the other, two regions which are formed consecutively are separated by other regions as the process is continued, and hence the expression C_{2n} may be negative in both these regions. This apparently introduces the possibility of the existence of a series in which some of the numbers are less than the corresponding numbers in the regular series, and in this case the number of internal ovals might be less than $\frac{1}{2}(n-1)(n-2)$. For example, if the last region of (b') and the first region of (c) are both negative, the series will contain a sequence of the form, $2t-7$, $(2t-4+\ldots)$ $2t-1$, $2t+3, \ldots$, (A), in which the numbers beginning with $2t-1$ are less by 2 than the corresponding numbers in the sequence $2t-7$, $2t-3$, $2t+1$, $2t+5 \ldots\ldots\ldots$ But the conditions for the existence of such a sequence as (A) demand the existence of a preceding sequence of the form $2l-9$, $2l-3$, $2l+1$, $2l+5$ instead of the regular sequence $2l-9$, $2l-5$, $2l-1$, $2l+3$. Thus the numbers in the series which precede such a sequence as (A) exceed by 2 the corresponding numbers in the regular series, and hence the numbers following the sequence will be the same as the corresponding numbers in the regular series.

Suppose such a sequence as $2p-7$, $(2p-4+\ldots)$, $2p-1$, $2p+3$ occurs in the passage from the set (a') to the set (b). The last region of (a') and the first region of (b) are both negative. Since the first region of (b) is negative, the last region of (a) must be positive. The last negative region of (a) then contains $2m-9$ ovals, and hence in the series of numbers, representing the groups of internal ovals there is a spring from $2m-9$ to $2m-3$, preceding the succession $2p-7$, $(2p-4+\ldots)$, $2p-1$, $2p+3$.

$$(a)\cdots\cdots\underrightarrow{\cdots\cdots+(b)-\cdots\cdots\cdots-(c)\cdots\cdots\cdots+}(d)\cdots\cdots\cdots(e)\cdots\cdots\cdots$$

$$(d')\cdots\cdots\cdots(c')+\cdots\cdots\underleftarrow{\cdots+(b')-\cdots\cdots\cdots}(a')-\cdots\cdots\cdots-$$

No such break can occur again in the series so long as the last region in each set considered is negative, for as a prerequisite to such a sequence as the above, the first region of some set must be negative and this is impossible unless the last region in the preceding contiguous set be positive. Let the first set whose last region is positive be (c). The last region of (b') is negative, hence the first region of (c') is positive. Therefore in the passage from (c) to (c') there is a jump over two annular regions, that is, from a group of $2l-9$ internal ovals to a group of $2l-3$. If the last region of (c') is positive, the series representing the groups of internal ovals goes on regularly except that in the 1st region of (*d*), which is negative since the last region of (*c*) is positive, the number of internal ovals is greater than it would have been had there been no change in the mode of generation. This regularity continues until the last region in some set is negative, and at this stage occurs again a sequence of the form $2r - 7$, $(2r - 4 + \ldots)$, $2r - 1$, $2r + 3 \ldots.$ In every case a sequence of the above form is preceded by a succession of the form $2l - 9$, $2l - 3$, $2l + 1$, $2l + 5$. Thus in the series representing the groups of internal ovals there may be numbers exceeding but none less than the corresponding numbers in the regular series. Hence since the series of numbers which represent the groups of internal ovals on curves derived by both modes of generation is either greater than or equal to the series obtained when the 1st mode of generation is used throughout, the number of internal ovals $\geqq \frac{1}{2}(n - 1)(n - 2)$.

The *third mode of generation* in which the new nested oval is taken as the generating oval, is not applicable to the generation of curves of degree lower than 8, for the 6^{ic} is the first curve whose generating oval can be a nested oval. Hence the first application of the process must be preceded by the use of the first or second mode of generation. Each application of the process reduces the number of nested ovals by 2. For suppose the new nested oval of C_{2m-2} cuts the ellipse; then on C_{2m} there is no *nested* oval representing this, and moreover none of the $4m - 4$ ovals arising from the segments of the ellipse and this generating oval encloses either the ellipse or the nested ovals lying inside the ellipse; that is, the nested oval contributed to the nest by C_{2m-2} disappears as such, and no new nested oval is added by C_{2m}. It is evident that since there is no nested oval among the new ovals formed at this stage, the third mode of generation cannot be applied twice in succession. Thus the different types of curves which are derived in part by the 3rd mode of generation are those obtained by the 1st and 3rd modes of generation or by the 2nd and 3rd, or by combinations of all three modes, or by combinations of these with the fourth. If the curves be derived by the 1st and 3rd *modes of generation*, let the 3rd mode be introduced for the first time for the generation of the curve C_{2m}. The new nested oval of C_{2m-2} lies inside the ellipse, and in the new annular region formed there are $2m - 8$ ovals, and in the region between the nested ovals and the ellipse there are $2m - 5$ ovals. The new nested oval of C_{2m-2} is to be taken as the generating oval. Hence in one of the $4m - 4$ regions bound by segments of E_2 and g_{2m-2} there are $2m - 5$ ovals, and therefore of the $4m - 4$ new ovals appearing on C_{2m}, one contains $2m - 5$ others, and this lies inside the ellipse. The conclusion is easily deduced as in the preceding case that the number of internal ovals on curves derived by the 2nd and 3rd modes of generation or by all three modes combined, is not less than $\frac{1}{2}(n-1)(n-2)$.

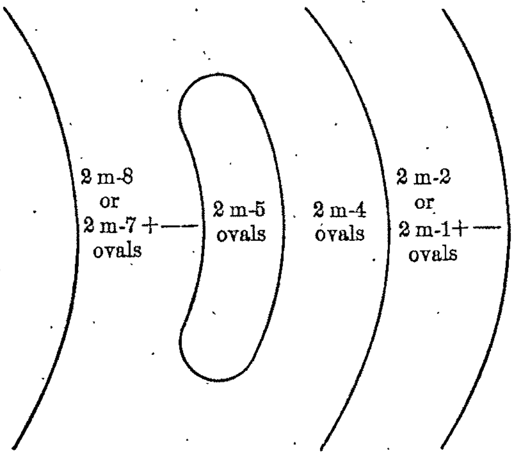

FIG. 8.

Although the 3rd mode of generation cannot be applied twice in succession, it need not be followed by the first or second, for the ellipse may be cut by the oval which contains $2m-5$ others. This introduces a *fourth mode of derivation*,

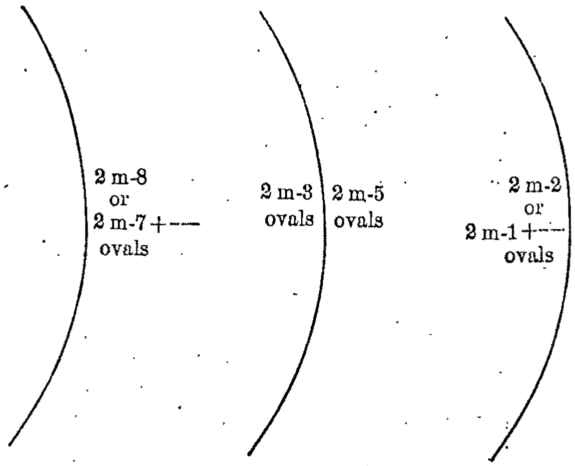

FIG. 9.

closely related however to the 3rd type, for though the generating oval is not one of the nested ovals, yet it does include others. It is not difficult to see that if the arrangement of the ovals in the annular regions formed at the stages

C_{2m-2}; C_{2m}, C_{2m+2}, C_{2m+4} on a curve derived by the 1st, 2nd and 3rd modes be represented by Fig. 8, the use of the 4th mode for the derivation of C_{2m+2} will modify the arrangement to that indicated by Fig. 9. That is, there is a combination in pairs of the four annular regions which would have appeared if instead of the 3rd and 4th modes of generation the one preceding the 3rd had been used; the first and third are united, and the second and fourth. If one region is negative, the other is positive, so that the same number of internal ovals is obtained as if the ovals were distributed in the four regions. Hence in this case, the introduction of the 4th mode of generation does not decrease the number of external ovals. If the 4th mode of generation be followed by the 3rd instead of the 1st or 2nd, then a somewhat different arrangement is produced. There are combined into one the five annular regions which would have appeared, if the mode by which C_{2m-2} was derived had been continued for the generation of C_{2m}, C_{2m+2}, C_{2m+4}, C_{2m+6}. Instead of the arrangement represented by Fig. 10, the result would be that indicated by Fig. 11.

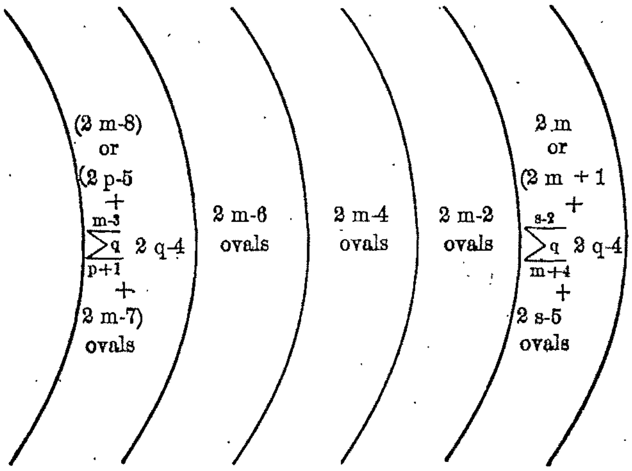

FIG. 10.

If all possible combinations be made of the four modes of generation, various arrangements of the circuits are obtained, but the investigation of the preceding combinations makes it evident that the number of internal ovals on the derived curve is equal to the number on a curve which is generated by the 1st and 2nd modes only and hence is not less than $\frac{1}{2}(n-1)(n-2)$.

The appearance of a curve derived by the 1st or 2nd mode of generation, or by a combination of the two, is only slightly modified by an occasional introduction of the 3rd mode of derivation; but an extensive use of the 3rd method gives a curve differing greatly in form from that derived by the 1st or 2nd mode alone, or by the 1st and 2nd together. For example, if for the derivation of the 8^{te} from the 6^{te} which is generated by the 1st mode of generation, the 3rd mode be employed, and for the derivation of curves of higher order the 4th and 3rd modes be used alternately, then no nest whatever is built up. The type of curves obtained is the same as that derived by the exclusive use of the 1st mode of generation in the Harnack process. The 8^{te} has one oval containing three, and the remaining ovals are external to one another; the 10^{te} has one oval

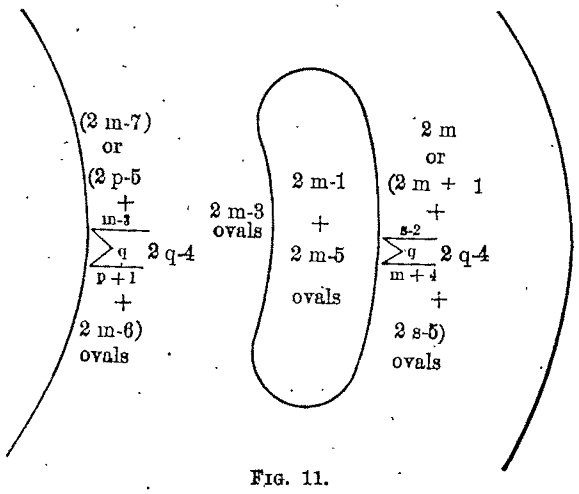

FIG. 11.

containing 6 and the remainder external to one another, and so on. If the 1st mode of generation is combined with the 3rd and 4th, but after the generation of the 8^{te} not applied twice in succession, there are derived in this way $m-2$ other types of curves of order $4m$, and $l-2$ other types of curves of order $2(2l-1)$, which agree with types obtained by the Harnack process. But no other types of the Harnack curves are derived by the use of the 2nd mode instead of the 1st or by the use of both the 1st and 2nd.

It is evident from the method of generation that the number of internal ovals can be diminished in the same manner as in the generation of curves by the Harnack process, and also that this diminution can in no case be accompanied by an increase in the number of external ovals. Therefore the number of external ovals is not greater than $n^2 + \frac{1}{2}(n-1)(n-2)$ even if the number of circuits falls short of the maximum number.

Curves Derived by the Modified Forms of the Hilbert Method.—The use of the *hyperbola* or *parabola* as auxiliary curve, since these can be projected into the ellipse, can certainly yield no type of $2n^{ic}$ differing from those obtained by the Hilbert process. Neither does the *degenerate conic.* This, however, requires a special proof. But when the pair of straight lines is substituted for the ellipse as the auxiliary curve, passage can be made at any time to the Harnack process, by the mere disregard of one of the lines, and a return to the original mode of derivation can be effected after the generation of any curve of even order. Thus new types of curves may arise by a combination of the two processes, but it can certainly be shown that no type of $2n^{ic}$ obtained has less than $\frac{1}{2}(n-1)(n-2)$ internal ovals.

If it could be shown that all non-singular curves with the maximum number of circuits can be generated by this method of small variation, the proof of the validity of the law for these remaining cases would establish it for all non-singular curves of even order. But as yet there is no formal proof that the list of such curves is exhausted by the types considered.

CONCLUSION.

There are several other forms in which the theorem can be stated that are of interest, either as facts resulting from the theorem if established in its preceding form, or as statements which may afford a better starting point for the proof of the theorem. A few of these equivalent forms are obtained by a consideration of the *Theory of the Characteristic*, which though apparently yielding no results toward the proof of the theorem, bears a most interesting relation to the problem. The theory as given by Kronecker* is purely algebraic; he proved that for any system of algebraic functions satisfying certain conditions, there exists a number derived algebraically which is invariant for that system. Dyck,† however, was led by a study of Kronecker's investigations to a *geometrical* definition of a characteristic number associated with a manifold, a number which is built up as the manifold itself is developed. He showed that if the manifold can be expressed algebraically, it can be developed by processes which also are capable of algebraic expression. The whole geometrical configuration thus

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*[*Ueber Systeme von Functionen mehrerer Variabeln*, Werke I, 1869, pp. 175–212, 213–226. *Ueber die Characteristik von Functionen Systemen*, Werke II, 1878, pp. 71–82.

†*Beiträge zur Analysis Situs*, I, II, III, Berichte der K. Sächs. Gesellschaft der Wissenschaften (Math. Phys. Classe) 1885, 1886, 1887. Math. Ann., vol. 32.

introduces a system of algebraic functions subject to certain conditious. Dyck proved that the characteristic number associated with the manifold can be derived from this system of functions and that the number is identical with the Kronecker characteristic of the system.

In Dyck's Theory the manifold is regarded as made up of elements, and to each element is assigned the characteristic $+1$. Whatever be the process of generation of the manifold,

(*a*) the appearance of a new element contributes $+1$ to the characteristic;

(*b*) the vanishing of an element contributes -1;

(*c*) the separation of an element into two pieces contributes $+1$;

(*d*) the joining of two elements, or of two parts of the same element, contributes -1.

For a one-dimensional manifold,—that is, a figure composed of lines,—the element is a broken piece of a curve. For a two-dimensional manifold, the element may be a part of a plane or of a surface that can be developed as a plane. The manifold suggested by any problem relating to the arrangement of the circuits on a plane curve, $f(xy) = 0$, of even order, with no singularities and with the maximum number of circuits, is obviously the two-dimensional manifold determined as the parts of the plane lying inside the curve,—that is, the parts of the plane where $f < 0$. The element is a piece of the plane bounded by a non-singular closed circuit (Fig. 12), and to this is assigned the characteristic $+1$. If the figure is initially non-existent, its characteristic is zero, and as the manifold is generated the characteristic is increased by unity as an element appears or separates into two, and is diminished by unity as an element vanishes, or as two elements or two parts of the same element unite. Therefore the characteristic of the manifold is equal to the sum of the characteristics of the separate parts which make up the manifold, and is independent of the mode of generation. Thus in the given manifold a part of the plane bounded by a single oval has the characteristic $+1$, whether it arises from a single element in the process of generation or from the union of several elements. Hence each external oval on the curve contributes to the manifold a region whose characteristic is $+1$. If two parts of such a piece of the plane unite and thus enclose in the midst of a region in which f is negative, a region where f is positive, their union is

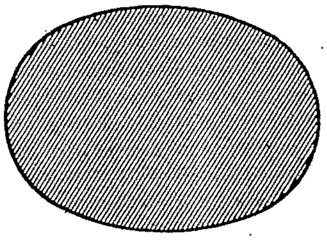


FIG. 12.

marked by -1 in the characteristic. But this is just the way in which an external oval makes its appearance. Hence the presence of each internal oval on the curve diminishes the characteristic of that piece of the plane to which it belongs by unity. The whole number of circuits on the $2n^{ic}$ is $n^2 + \frac{1}{2}(n-1)(n-2)$, and hence according to the theorem which ascribes the minimum limit $\frac{1}{2}(n-1)(n-2)$ to the number of internal ovals, the characteristic cannot exceed $n^2 + \frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-1)(n-2)$ or n^2, if the maximum number of circuits is present. It has been noted that a decrease in the number of internal ovals cannot be accompanied by an increase in the number of external ovals. Hence the characteristic cannot be greater than $n^2 + \frac{1}{2}(n-1)(n-2)$ even if the number of branches is not the maximum number.

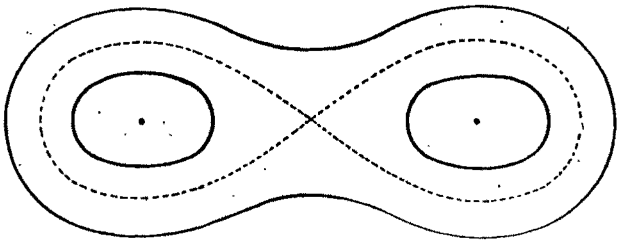


FIG. 18.

The process of generation of such a manifold can be expressed analytically by considering the manifold as one of a singly infinite system of manifolds. Let the curve $f=0$ be obtained as one of the pencil $f=\lambda$. Then the region where f is negative decreases as λ decreases, and for some value of λ sufficiently near $-\infty$, the curve disappears altogether. The characteristic of the corresponding manifold is zero. Let λ increase from this value. The curve makes its appearance as an isolated point spreading into a circuit as λ continues to increase. Other circuits also may come into existence in the same way. Thus an isolated point gives rise to a part of the plane and hence has the value $+1$ for the characteristic. Two circuits may unite, producing a node (Fig. 13) and thus join two pieces of the plane together, or a single circuit may cut itself (Fig. 14), and so unite parts of the same region of the plane. In either case the node is marked by -1 in the characteristic. As λ increases, an internal oval shrinks until it becomes an isolated point and then passes out of existence, or it may cut itself in such a way that at the next stage it breaks into two internal ovals. In all cases the isolated points which present themselves in the region $f<0$ contribute $+1$ to the characteristic, and the nodes -1. The singular points of the system are given by $f_1 = 0, f_2 = 0$, and are nodes with real tangents or isolated points according as $\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix}$ is negative or positive. If the notation $\left[\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} \right]$ be used to represent $+1, 0, -1$ according as the determinant is positive, zero, or negative, the characteristic of the manifold can be written $X = \Sigma \left[\begin{vmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{vmatrix} \right]$, the summation taken over all points $f_1 = 0, f_2 = 0, f < 0$; and Dyck shows that this is the same as the Kronecker characteristic for the system of functions f, f_1, f_2. Since the characteristic cannot exceed n^2 if $f = 0$ be a curve (order $2n$) with the maximum number of circuits, it follows that the number of isolated points passed over in the system $f = \lambda$ from $\lambda = -\infty$ to $\lambda = 0$ exceeds the number of nodes by a quantity less than, or equal to n^2. And if $f = 0$ has not the maximum number of circuits, the excess of the number of isolated points over the number of nodes in the region $f < 0$ is either less than or equal to $n^2 + \frac{1}{2}(n-1)(n-2)$.

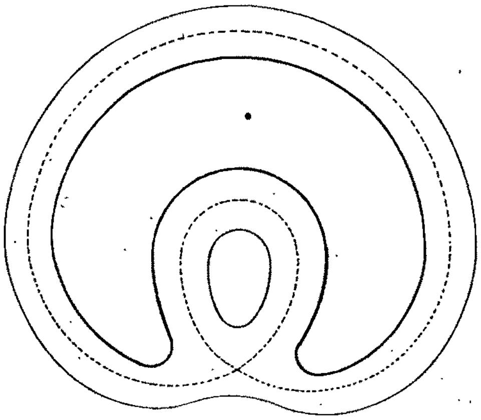


FIG. 14.

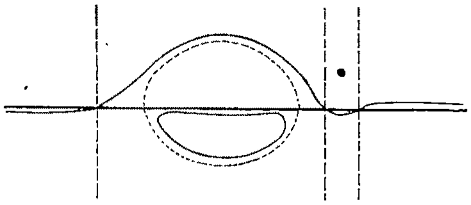
The relations between the critic centres of the pencil $f = \lambda$ thus obtained in applying the Theory of the Characteristic to an interpretation of the problem are interesting, but afford no clue to the solution. There is even some indication that the theory is not the most promising instrument of proof, for this is applicable to curves of even order only, and though the theorem on the minimum limit of the number of internal ovals is stated for these curves alone, it most probably can be extended to include curves of odd order also. It has been seen that on a non-singular $2n + 1^{ic}$ with the maximum number of branches the ovals may lie each outside the others; but even in this case they may satisfy the definition of "internal ovals," for the odd circuit divides the plane into two regions both infinite, in one of which C_{2n+1} is positive, and in the other, negative. And as a matter of fact, no $2n+1^{ic}$ with fewer than $\frac{1}{2}(n-1)(n-2)$ interval ovals presents itself directly by either of the two processes of generation discussed.

There are still other forms in which the theorem can be stated, the most interesting of which is perhaps one relating to the number of regions into which the plane is divided by the curve. It has been shown that if the maximum number of branches is present, then the curve must have at least $\frac{1}{2}(n-1)(n-2)$ *internal* ovals, and whether the number of branches present is the maximum or less than the maximum, the curve can not have more than $n^2+\frac{1}{2}(n-1)(n-2)$ *external* ovals. By a similar line of reasoning it can be proved that whatever the number of circuits, the number of internal ovals can not exceed $n^2+\frac{1}{2}(n-1)(n-2)-1$; and hence if the maximum number of circuits is present the number of external ovals can not fall below $\frac{1}{2}(n-1)(n-2)+1$.

Therefore if the maximum number of branches is present,
the number of regions in which the expression C_{2n} is positive $\geqq \frac{1}{2}(n-1)(n-2)+1$, and the number of regions in which C_{2n} is negative $\geqq \frac{1}{2}(n-1)(n-2)+1$; and whatever the number of circuits,
the number of regions in which C_{2n} is positive $\leqq n^2+\frac{1}{2}(n-1)(n-2)$ and the number of regions in which C_{2n} is negative $\leqq n^2+\frac{1}{2}(n-1)(n-2)$. From these statements, it seems that any limitation on the arrangement of the circuits is of a dual nature; and it is worthy of note that no modification of these statements is necessary, if the sign of C_{2n} be so chosen that it is negative at infinity.

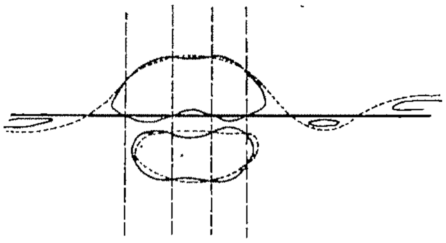
The statement of the theorem in this last form suggests that there may be some underlying relation to the theory of Multiply-connected Surfaces.

In the figures of Plates I and II the curves are much distorted, inflexions being inserted where none exist, in order to bring the figures within the scale of the paper. The figures represent the distribution of the ovals of the curves accurately only with respect to the *number* in different regions of the plane. In the drawing of the figures representing curves of the type, $C_{2n} \equiv C_{2n-2} \cdot E_2 + \delta \prod_{i=1}^{i=2n} l_i = 0$, the straight lines l_i are omitted.



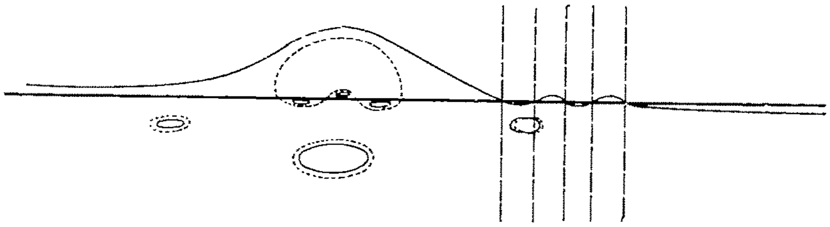
$$\underline{C_3} \equiv \underset{\cdots}{C_2} \cdot \underline{v} + \delta \prod_{i=1}^{i=3} \underline{l_i} \,\underline{=}\, \underline{0}.$$

FIG. 1.



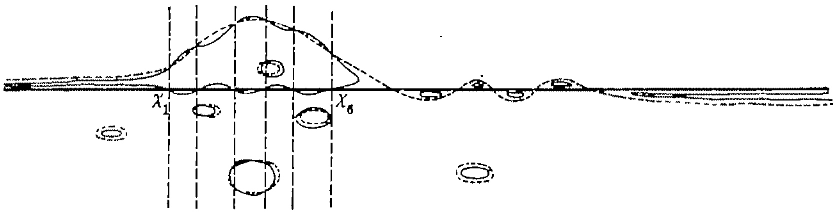
$$\underline{C_4} \equiv \underset{\cdots}{C_3} \cdot \underline{v} + \delta \prod_{i=1}^{i=4} \underline{l_i} \,\underline{=}\, \underline{0}.$$

FIG. 2.



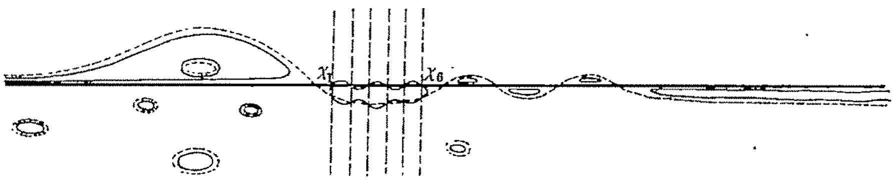
$$\underline{C_5} \equiv \underset{\cdots}{C_4} \cdot \underline{v} + \delta \prod_{i=1}^{i=5} \underline{l_i} \,\underline{=}\, \underline{0}.$$

FIG. 3.



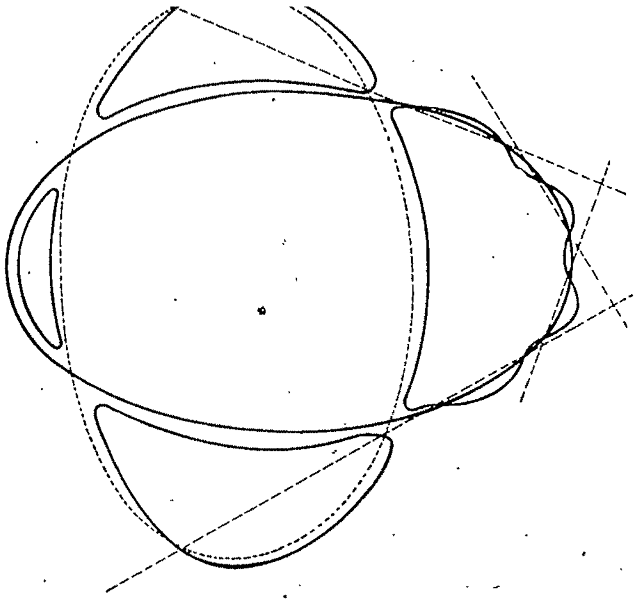
$$\underline{C_6} \equiv \underset{\cdots}{C_5} \cdot \underline{v} + \delta \prod_{i=1}^{i=5} \underline{l_i} \,\underline{=}\, \underline{0}.$$

FIG. 4.



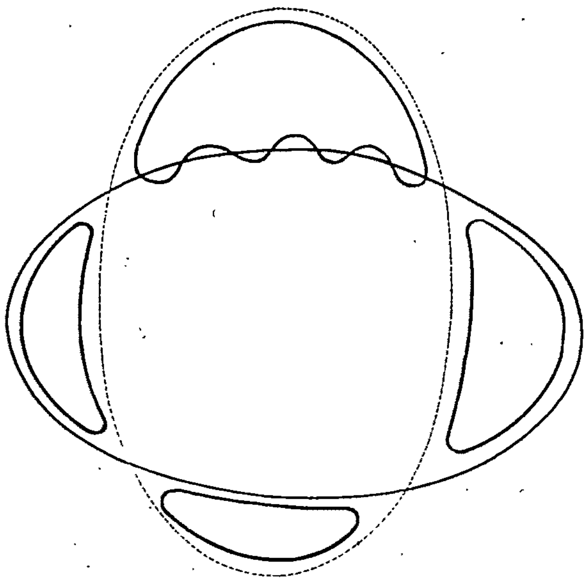
$$\underline{C_6} \equiv \underset{\cdots}{C_5} \cdot \underline{v} + \delta \prod_{i=1}^{i=6} \underline{l_i} \,\underline{=}\, \underline{0}.$$

FIG. 5.



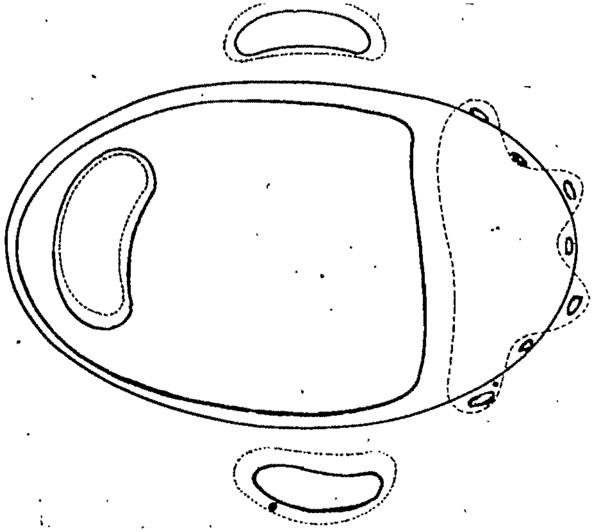
$$\underline{C_4} \equiv \underset{\cdots}{C_2} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=4} l_i = 0.$$

FIG. 1.



$$\underline{C_4} \equiv \underset{\cdots}{C_2} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=4} l_i = 0.$$

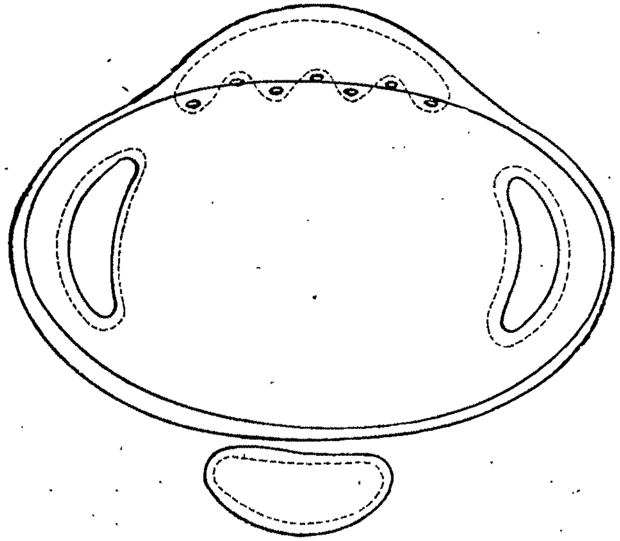
FIG. 2.



$$\underline{C_6} \equiv \underset{\cdots}{C_4} \; \underline{E_2} + \delta \prod_{i=1}^{i=6} l_i = 0.$$

C_6 from C_4 by 1st mode of generation.

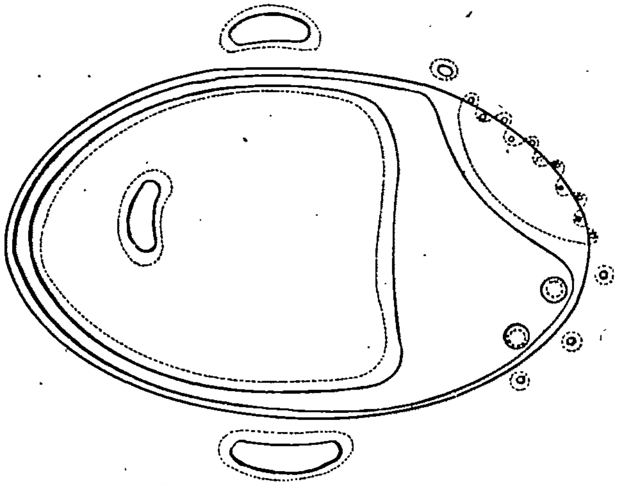
FIG. 3.



$$\underline{C_6} \equiv \underset{\cdots}{C_4} \; \underline{E_2} + \delta \prod_{i=1}^{i=6} l_i = 0.$$

C_6 from C_4 by 2nd mode of generation.

FIG. 4.

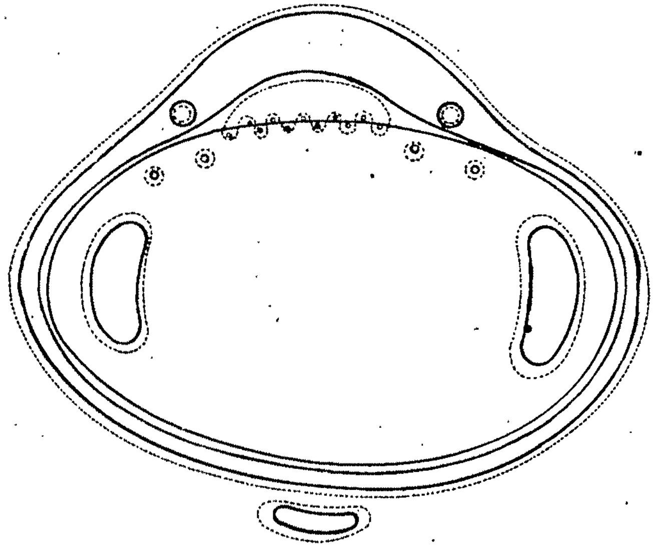


$$\underline{C_8} \equiv \underset{\cdots}{C_6} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=8} l_i = 0.$$

C_8 from C_6,

C_6 from C_4 by 1st mode of generation.

FIG. 5.



$$\underline{C_8} \equiv \underset{\cdots}{C_6} \cdot \underline{E_2} + \delta \prod_{i=1}^{i=8} l_i = 0.$$

C_8 from C_6,

C_6 from C_4 by 2nd mode of generation.

FIG. 6.